Statistical Performance of Gaussian ADC Histogram Test

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Abstract – In this paper, the statistical properties of the Gaussian Histogram Test (GHT) are theoretically evaluated and compared to the corresponding Cramér-Rao Lower Bound (CRLB). In particular, it is shown that the GHT is asymptotically unbiased and efficient. Finally, the GHT is compared to the Sinewave Histogram Test (SHT) one.

Keywords – Gaussian Histogram Test, Sinewave Histogram Test, Cramér-Rao Lower Bound.

I. INTRODUCTION

Analog to Digital Converter (ADC) testing may be an expensive and time consuming process, whose costs grow quickly with the converter resolution and are a consistent part of manufacturing costs [1]. Thus, an effective testing procedure may positively affect a manufacturing activity, allowing the reduction of both production times and costs. The ADC characterization is usually performed by measuring the ADC transition levels, which are used to derive figures of merit such as Integral and Differential Non-Linearities [2][3]. Various testing procedures have been published about this topic, one of the most popular being the Sinewave Histogram Test, which requires the analysis of the ADC output codes when a sinewave stimulus is applied [1-4]. However, in various application fields, e.g. Digital Communications, the converted signals exhibit Gaussian amplitude-distributions [5]. Hence, ADC stimuli different from the sinewave have been considered for characterizing the performance of both the ADCs and the systems in which these devices are embedded [5][6]. This paper is focused on the performance of an Histogram Test based on white Gaussian input signals. First, the bias and variance of the estimator have been theoretically evaluated, extending the results provided in [6]. Then, in order to evaluate the statistical efficiency of the Gaussian Histogram Test (GHT), the estimator variance has been compared to the corresponding Cramér-Rao Lower Bound (CRLB), which has also been derived in closed form. In particular, it is shown that the GHT is asymptotically unbiased and statistically efficient when the size of the stimulus sample is large. Finally, the GHT has been compared to the Sinewave Histogram Test (SHT), with respect to the estimator variance.

II. THE GAUSSIAN HISTOGRAM TEST

The GHT is performed by feeding an ADC with a white Gaussian stimulus, whose amplitude is properly matched to the ADC dynamic range [6]. In particular, the stimulus amplitude should be large enough to possibly excite all the ADC output codes. Then, for each ADC output code, the cumulative histogram C_k, that is the number of output samples not exceeding the corresponding output code, is evaluated. This methodology relies on the relative frequency approach to the probability theory, which states that a cumulative histogram, normalized to the number of converted samples N, tends to the probability p_k that the ADC output does not exceed the corresponding code when N becomes large. This probability is given by

$$p_k = F_x(T_k), \qquad k = 0,\ldots,2^m - 1, \tag{1}$$

where T_k is the k-th ADC transition level of an m-bit ADC, and $F_x(\cdot)$ is the probability distribution function of the stimulus x. Hence, by inverting (1), the transition levels can be obtained as a function of the probability estimator distribution function $\hat{p}_k = C_k/N$. In particular, when the stimulus is Gaussian, the estimated transition levels $\hat{T}_k$ are given by

$$\hat{T}_k = g(\hat{p}_k) = \begin{cases} -FS, & \hat{p}_k = 0 \\ \sigma\sqrt{2}\,erf^{-1}(2\hat{p}_k - 1) + \mu, & 0 < \hat{p}_k < 1, \\ FS, & \hat{p}_k = 1 \end{cases} \tag{2}$$

$k = 0,\ldots,2^m - 1$

where FS is the ADC Full Scale, σ and μ are the standard deviation and the mean value of the Gaussian stimulus respectively, that can be estimated using experimental data [2], and erf(·) is the error function [6]. The estimator has been redefined in $\hat{p}_k=0$ and $\hat{p}_k=1$ because for such values the inverse error function is singular.

The bias $b_{\hat{T}_k}$ and the variance $\sigma^2_{\hat{T}_k}$ of the estimator $\hat{T}_k$ can be evaluated by considering that $\hat{p}_k$ is a discrete random variable, thus obtaining:

$$b_{\hat{T}_k} = E\left[\hat{T}_k\right] - T_k = \sum_{n=0}^{N} g\left(\tfrac{n}{N}\right) \Pr\left\{\hat{p}_k = \frac{n}{N}\right\} - T_k \tag{3}$$

$$\sigma^2_{\hat{T}_k} = \sum_{n=0}^{N} \left(g\left(\tfrac{n}{N}\right) - E\left[\hat{T}_k\right]\right)^2 \Pr\left\{\hat{p}_k = \frac{n}{N}\right\}, \tag{4}$$

where $\Pr\left\{\hat{p}_k = \frac{n}{N}\right\} = \binom{N}{n} p_k^n (1-p_k)^{N-n}$, k=0,…,2^m-1, and n=0,…,N.

Equations (3) and (4) show an excellent agreement with simulation results. However, due to the factorial terms in the binomial coefficient, they can not be easily applied for large values of N. Consequently, an approximated asymptotic model has been derived, which conveniently describes $b_{\hat{T}_k}$ and $\sigma^2_{\hat{T}_k}$ under such condition. In fact, as the mean value $\mu_{\hat{p}_k}$ and variance $\sigma^2_{\hat{p}_k}$ of a binomial random variable $\hat{p}_k$ are respectively given by

$$\mu_{\hat{p}_k} = p_k, \qquad \sigma^2_{\hat{p}_k} = \frac{p_k(1-p_k)}{N}, \tag{5}$$

we notice that, when N is large, the distribution of $\hat{p}_k$ is concentrated near $\mu_{\hat{p}_k}$. Hence, by applying a Taylor series expansion of $g(\cdot)$ about the expected value of $\hat{p}_k$, the bias $b_{\hat{T}_k}$ and the variance $\sigma^2_{\hat{T}_k}$ of $\hat{T}_k$ can be evaluated as a function of $\mu_{\hat{p}_k}$ and $\sigma^2_{\hat{p}_k}$, according to the following [7]:

$$b_{\hat{T}_k} \cong g''(\mu_{\hat{p}_k}) \cdot \frac{\sigma^2_{\hat{p}_k}}{2}, \tag{6}$$

$$\sigma^2_{\hat{T}_k} \cong g'(\mu_{\hat{p}_k})^2 \cdot \sigma^2_{\hat{p}_k}, \qquad \mu_{\hat{p}_k} \in \left]0,1\right[, \qquad k = 0,\ldots,2^m-1, \tag{7}$$

where $g'(\cdot)$ and $g''(\cdot)$ are the first and second derivatives of $g(\cdot)$, respectively. By expressing the derivatives of $\mathrm{erf}^{-1}(\cdot)$ in terms of the derivatives of the error function (see appendix A), $b_{\hat{T}_k}$ and $\sigma^2_{\hat{T}_k}$ can be approximately expressed as a function of the corresponding transition level, as follows

$$b_{\hat{T}_k} \cong \frac{\sigma^2 (T_k-\mu)}{2N} \frac{\Phi\left(\frac{T_k-\mu}{\sigma}\right)\left(1-\Phi\left(\frac{T_k-\mu}{\sigma}\right)\right)}{\phi\left(\frac{T_k-\mu}{\sigma}\right)^2}, \qquad k = 0,..,2^m-1 \tag{8}$$

$$\sigma^2_{\hat{T}_k} \cong \frac{\sigma^2}{N} \frac{\Phi\left(\frac{T_k-\mu}{\sigma}\right)\left(1-\Phi\left(\frac{T_k-\mu}{\sigma}\right)\right)}{\phi\left(\frac{T_k-\mu}{\sigma}\right)^2}, \qquad k = 0,..,2^m-1 \tag{9}$$

where $\Phi(\cdot)$ and $\phi(\cdot)$ are the probability distribution and the probability density functions of a Gaussian random variable, with zero mean and unitary standard deviation respectively. Equation (9) shows that both the GHT bias and variance are minimized when the stimulus mean value is close to the ADC transition level. This is related to the amplitude distribution of the Gaussian stimulus, which excites more often transition levels close to its mean value. Finally, (8) shows that the GHT is asymptotically unbiased.

III. CRAMÉR-RAO LOWER BOUND ON TRANSITION LEVEL ESTIMATORS WITH GAUSSIAN STIMULUS

The statistical efficiency of an estimator can be evaluated by comparing its variance to the corresponding CRLB, which, for the considered case, can be theoretically calculated by exploiting both the characteristics of the estimated parameters and the statistical properties of the stimulus. In particular, it can be shown (see Appendix B) that the CRLB on the variance of unbiased estimators of an ADC transition level is:

$$CRLB(T_k) = \frac{\sigma^2}{N} \frac{\Phi\left(\frac{T_k-\mu}{\sigma}\right)\left(1-\Phi\left(\frac{T_k-\mu}{\sigma}\right)\right)}{\phi\left(\frac{T_k-\mu}{\sigma}\right)^2}, \qquad k = 0,..,2^m-1 \tag{10}$$

While (11) applies for unbiased estimators, (3) and (8) show that the GHT is a biased estimator. Consequently, the GHT variance should be compared to the Cramér-Rao Lower Bound on the variance of estimators affected by a given bias $b_{\hat{T}_k}$, that is [8]

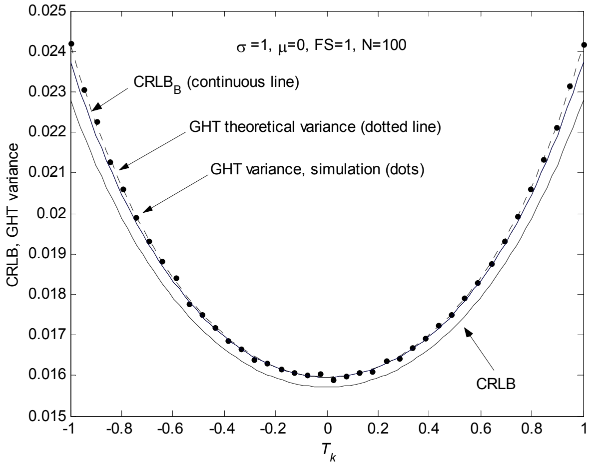

Fig. 1: Theoretical (Eq.(4)) and simulated variance of the GHT estimator, and CRLB on the variance of biased and unbiased estimators. Records of N=100 samples.

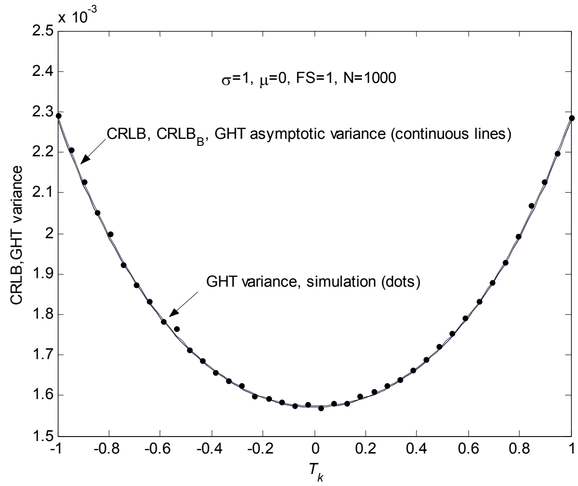

Fig. 2: Theoretical (Eq.(9)) and simulated variance of the GHT estimator, and CRLB on the variance of biased and unbiased estimators. Records of N=1000 samples.

$$CRLB_B(T_k) = \left(1 + b'_{\hat{T}_k}\right)^2 CRLB(T_k), \tag{11}$$

where $b'_{\hat{T}_k}$ is obtained by differentiating $b_{\hat{T}_k}$ with respect to T_k.

Figure 1 summarizes the analysis results, obtained for records of N=100 samples of a zero mean Gaussian stimulus, with a unitary variance. Such a low value has been chosen to show the effect of the estimator bias. The GHT variance, obtained both theoretically and by means of simulations, is reported as a function of the value T_k of the transition level. As expected, the exact model (4) shows a very good agreement with simulation results. Furthermore, in order to evaluate the GHT statistical efficiency, both CRLB and CRLB$_B$, given by (10) and (11) respectively, are plotted. It can be observed that the estimator variance is very close to the CRLB$_B$ curve, showing that for low values of N the GHT is an almost efficient estimator. Moreover, by comparing the CRLB curve to the CRLB$_B$ curve, it results that when N is small the effects of bias can not be neglected. However, by analyzing (8) and (11), it can be observed that, as $b'_{\hat{T}_k}$ tends to zero like $1/N$, the difference between CRLB and CRLB$_B$ vanishes like $1/N^2$. Consequently, when N is large, CRLB$_B$ tends asymptotically to the CRLB obtained for unbiased estimators, given by (10). In this regard, it is worth of notice that (10), coincides with (9), which models the asymptotic expression for the GHT variance. Hence, the GHT is asymptotically unbiased and efficient. These results have been confirmed by meaningful simulations, carried out for large values of N. In particular, the GHT variance, obtained using records of N=1000 samples, is reported in Fig. 2, together with the results given by (9) and (11). As expected, the asymptotic model (9) shows a very good agreement with simulation results, and the difference between (9) and (11) is negligible.

IV. COMPARISON BETWEEN GAUSSIAN AND SINEWAVE HISTOGRAM TEST ACCURACY

Fig. 3, obtained for record length N equal to 100 samples, reports the variance of the GHT, given by (9), and the variance of the SHT, given by (4) of [10] reported here for the sake of completeness,

$$\sigma^2_{SHTk} \cong \alpha_k(1-\alpha_k)\frac{\pi^2}{N^2}[A^2 - \overline{T}_k^2] + \frac{1.78}{N}\sigma_w\sqrt{A^2 - \overline{T}_k^2}\ , \tag{12}$$

$$\overline{T}_k = T_k - d,\ \ \alpha_k = \left\langle \frac{2\psi_k}{\Delta\varphi} \right\rangle,\ \ \Delta\varphi = 2\pi / N,\ \ \psi_k = \arccos\left(-\frac{\overline{T}_k}{A}\right)$$

where A is the sinewave amplitude, d is the sinewave offset and σ_w is the standard deviation of an additive Gaussian noise superimposed to the sinewave stimulus. In order to achieve comparable results, both the Gaussian and the Sinewave stimuli have unit power. Furthermore, the SHT performance has been considered both for a noiseless stimulus and for noisy sinewave, with a signal-to-noise ratio (SNR) of 10dB. It can be observed that when the stimulus is a noiseless sinewave, the SHT is more efficient than the GHT, because of the deterministic nature of the signal. Moreover, the performance gap between the noiseless SHT and the GHT widens quickly when N is increased. In fact, the left addendum in (12), which models the deterministic contribution to the SHT variance, vanishes like $1/N^2$, while (9) tends to zero like $1/N$. However, when the SHT is performed with a noisy sinewave, for large

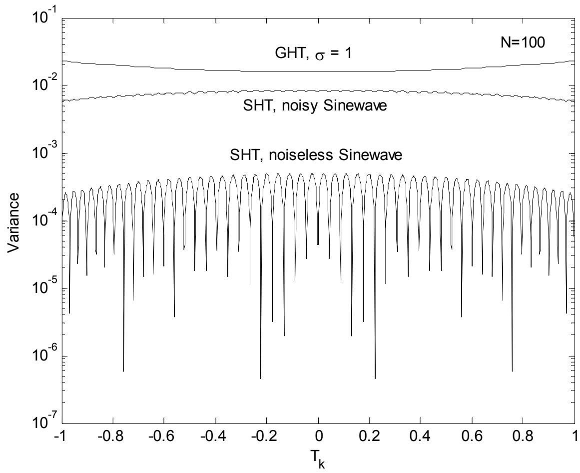

Fig. 3: Comparison of the variance of the GHT and the SHT, for a record length of N=100 samples. Both the Gaussian and the Sinusoidal stimuli have unitary power.

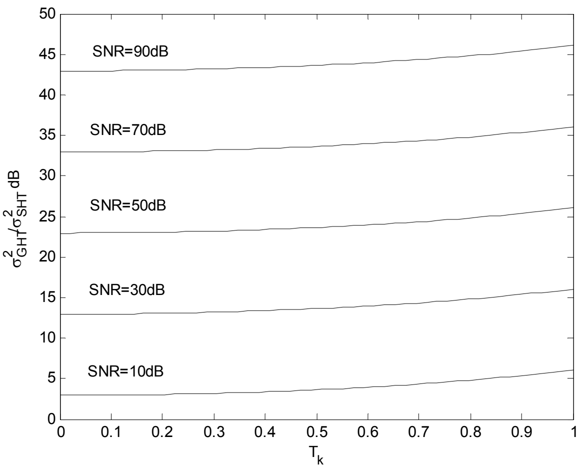

Fig. 4: Performance difference between the GHT and the SHT.

values of N the SHT variance is dominated by the random noise contribution, which is modeled by the right addendum of (12) and tends to zero like $1/N$. Consequently, the performance difference between the GHT and the SHT does not depend on the record length. Fig. 4 reports the ratio between the GHT variance σ^2_{GHT} and the SHT variance σ^2_{SHT}, obtained for large values of N, as a function of the transition level T_k. Various curves are reported, obtained for various SNRs of the SHT stimulus, which show that the performance gap between the GHT and SHT is mostly influenced by the sinewave SNR. In particular, the difference has a minimum in T_k=0, because under such condition σ^2_{GHT} is minimized and σ^2_{SHT} is maximized. This value can be estimated by the following

$$\min\left(\frac{\sigma^2_{GHT}}{\sigma^2_{SHT}}\right) = \frac{\pi}{2\sqrt{2}\cdot 1.78}\sqrt{SNR}, \quad SNR = \frac{A^2/2}{\sigma^2_w}, \tag{13}$$

which can be derived by putting T_k=0 in (9) and (12).

V. CONCLUSIONS

The performance of the Gaussian Histogram Test has been analyzed, with respect to the estimator statistical properties. Closed form results have been derived, which correctly model both the estimator bias and variance and the corresponding Cramér-Rao lower bound. In particular, it has been proved that the GHT provides almost efficient and unbiased estimators when a Gaussian stimulus is used. Finally, the GHT has been compared to the SHT with respect to the estimator variance, showing that when the record length becomes large the performance difference is mostly affected by the SNR of the SHT stimulus.

APPENDIX A - *Bias and variance of the GHT*

The derivatives of $g(\cdot)$ can be expressed in terms of the derivatives of its inverse function $f(\cdot)=g^{-1}(\cdot)$, according to the following

$$g'(x) = \frac{1}{f' \circ f^{-1}(x)} \tag{A.1}$$

$$g''(x) = -\frac{f'' \circ f^{-1}(x)}{\left(f' \circ f^{-1}(x)\right)^3}, \tag{A.2}$$

where $\circ$ is the composition of function operator and $f(x) = g^{-1}(x) = 1/2 + erf(\frac{x-\mu}{\sigma})$. According to (5) and (6), the GHT variance is expressed by

$$\sigma^2_{\hat{T}_k} \cong g'(p_k)^2 \cdot \frac{p_k(1-p_k)}{N}, \tag{A.3}$$

$$b_{\hat{T}_k} \cong g''(p_k) \cdot \frac{p_k(1-p_k)}{2N}, \tag{A.4}$$

where $p_k = f(T_k) = \Phi\left(\frac{T_k-\mu}{\sigma}\right)$. By replacing p_k with $f(T_k)$ in (A.1) and (A.2), the following are obtained,

$$g'(p_k) = \frac{1}{f'(T_k)} = \frac{\sigma}{\phi\left(\frac{T_k-\mu}{\sigma}\right)} \tag{A.5}$$

$$g''(p_k) = -\frac{f''(T_k)}{f'(T_k)^3} = \frac{T_k-\mu}{\phi\left(\frac{T_k-\mu}{\sigma}\right)^2} \tag{A.6}$$

Finally, by substituting (A.5) and (A.6) in (A.3) and (A.4), (8) and(9) are obtained.

APPENDIX B - *Cramér-Rao Lower Bound on the estimation of ADC transition levels*

If an unbiased estimator is given by a function $\tau(\theta)$ of a parameter θ in a given statistical model, the corresponding CRLB is obtained as

$$CRLB(\theta)=\frac{\tau'(\theta)^2}{F}, \qquad \text{(B.1)}$$

where F is the Fisher information, and $\tau'(\theta)$ is the first derivative of $\tau(\theta)$ with respect to θ [8][9]. By assuming $\theta=T_k$, we have that $\tau'(\theta)=1$ and the CRLB can be expressed as $1/F$. The estimation of a transition level T_k can be described by a Bernoulli model, because the stimulus is a sequence of independent Gaussian random variables, and all the ADC input samples have the same probability p_k of not exceeding T_k, given by (1). As Bernoulli models are regular, the Fisher information associated to a sample of size N is N times the Fisher information of an unitary sized sample, that is [8]

$$CRLB=\frac{1}{NF_1}, \qquad \text{(B.2)}$$

where F_1 is the Fisher information of a unitary sized sample. For such a sample, only two events are possible, depending if the sample does or does not exceed the threshold., characterized respectively by the probabilities of occurrence p_{0k} and p_{1k}, where

$$p_{0k}=1-\Phi\left(\frac{T_k-\mu}{\sigma}\right), \qquad p_{1k}=1-p_{0k}=\Phi\left(\frac{T_k-\mu}{\sigma}\right) \qquad \text{(B.3)}$$

Consequently, F_1 can be expressed as

$$F_1=\frac{1}{p_{0k}}\left(\frac{\partial p_{0k}}{\partial T_k}\right)^2+\frac{1}{p_{1k}}\left(\frac{\partial p_{1k}}{\partial T_k}\right)^2=$$
$$=\frac{1}{\sigma^2}\phi\left(\frac{T_k-\mu}{\sigma}\right)^2\left(\frac{1}{1-\Phi\left(\frac{T_k-\mu}{\sigma}\right)}+\frac{1}{\Phi\left(\frac{T_k-\mu}{\sigma}\right)}\right), \qquad \text{(B.4)}$$

and by substituting (B.4) in (B.2), (10) is obtained.

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