NEW APPROACH TO THEORY OF SIGMA-DELTA ANALOG-TO-DIGITAL CONVERTERS

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Abstract –The main point of new approach to the theory of sigma-delta modulation is application of a discrete two-values distribution law for a quantization noise instead of a uniform distribution law accepted before. Due to new approach, the variance (standard deviation in the square) of the quantization noise becomes dependent on input signal by parabolic function. The noise vs. frequency is found for different input signals taking into account all frequency range from zero to half of sampling frequency. Using dependence of standard deviation on input signal and frequency, different characteristics of sigma-delta modulation can be predicted, including *SNR*. Difference between analytical and simulation results for new theory can be driven to any small value.

1. Introduction

A quantization noise of sigma-delta modulation was supposed to be described by a uniform distribution law as in the case of a quantization error for any ADC [1-3]. Though this approach was not evident [1] and gave gross errors [4], it was accepted in all papers known to the authors. As a result, the quantization noise for a modulator with a simple comparator was found as an additive error with a standard deviation $\sigma = \frac{1}{\sqrt{3}} \approx 0.578$. The comparator was represented as an amplifier followed by a noise voltage source. The quantization noise was supposed to have constant spectral density ("white noise"), though discrete spices depending on input signal were mentioned [1]. Definition of equivalent comparator gain η was given using simulation [1] depending on signal value. For low frequency formula for *SNR* was found [1] as:

$$SNR = 10\log\frac{3a^2(2n+1)R^{2n+1}}{2\alpha^2\pi^{2n}} \tag{1}$$

where n is the modulator order, a is the input signal amplitude, α is the gain of noise transfer function, R is the oversampling ratio equal to ratio of comparator sampling frequency f_S to sampling frequency at the output of the digital filter [2, 3]. The latter frequency is usually supposed to be equal to double maximum frequency of input signal f. The value of α was usually supposed equal to 1 [1-4]. The standard deviation can be found from (1) as:

$$\sigma(f) = \frac{\alpha\pi^n}{\sqrt{3(2n+1)R^{2n+1}}} \tag{2}$$

For the first order modulator $n = 1$, $\sigma(f) \approx 1.05\alpha / \sqrt{R^3}$, $SNR \approx 10\log 0.456a^2R^3/\alpha^2$.

2. New approach to modelling of sigma-delta modulator

The quantization noise of the sigma-delta modulator is a variable even for constant input direct current signal. This variable is represented at the output of the modulator Y by one of two possible values: V_{REF} or $-V_{REF}$. The first value is usually shown as +1, the second value is –1. Then the input signal range is $-1 \le X \le 1$. For quantization noise analysis, all modulator components (switches, integrators, a comparator) are supposed to be ideal. Then the expectation of a modulator output must be equal to input signal X and only two levels of error take place: $\Delta_1 = -1-X$ for $Y = -V_{REF}$ and $\Delta_2 = 1-X$ for $Y=$

V_{REF}. At any random time moment, the realization of Δ_1 or Δ_2 is a random case. Probability of the first case is $P_1 = 0.5(1-X)$ while probability of the second case is $P_2 = 0.5(1+X)$. This random quantity is described by discrete two-value distribution law (not by uniform one!) that is shown in Fig. 1.

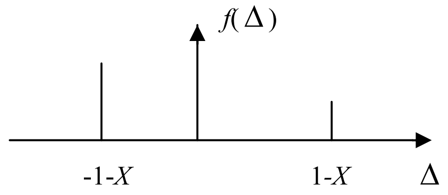

Figure 1. Distribution law of the quantization noise for a sigma-delta modulator

Two δ - functions (Dirac functions) are shown in Fig. 1 representing P_1 and P_2 correspondingly. The expectation of the probability law:

$$M(\Delta) = -(1+X)0.5(1-X) + (1-X)0.5(1+X) = 0 \tag{3}$$

Variance of the probability law:

$$\sigma^2 = (-1-X- M(\Delta))^2 0.5(1-X) + (1-X- M(\Delta))^2 0.5(1+X) = 1-X^2 \tag{4}$$

Equations (3) and (4) are true for any order of a modulator, both for a continuous-time one and a discrete-time modulator. The first order continuous-time modulator is analyzed below in default. In opposite to many previous papers (see section 1), the variance given by (4) depends on input signal. It is going to zero when input signal is closed to ends of the range ($X=\pm 1$). For $X = 0$ it is equal to 1 and three times more in comparison with the value used before. Different ways can be used to check (4) by simulation. Let's consider a sequence of N pulses at the modulator output including N_1 positive and N_{-1} negative pulses. Evaluations (5) and (6) can describe expectation of input signal and standard deviation correspondingly:

$$\breve{X} = (N_1 - N_{-1}) / N \tag{5}$$

$$\breve{\sigma} = \sqrt{1 - \breve{X}^2} \tag{6}$$

If N corresponds to the period of pulses at the output of the modulator, then $\breve{X} = X$ and $\breve{\sigma} = \sigma$. Such situation takes place if, according to (5), product of N by $0.5(1+X)$ gives the entire number of N_1 (the entire number of $N_{-1} = N - N_1$ correspondingly). Such signals are named in papers [1, 2] as "idle tones" without detailed analysis. Maximum relative deflections between X and $\breve{X}$, σ and $\breve{\sigma}$ equal correspondingly to:

$$\delta_X = \pm \frac{2}{NX} \tag{7}$$

$$\delta_\sigma \approx \frac{2(\mp X - N^{-1})}{N(1 - X^2)} \tag{8}$$

Both relative errors can be made as low as necessary if the value N is large enough. If X increases, then more value N is necessary for given relative error. Equation (4) corresponds to all frequency range of noise from 0 to $0.5f_S$. The latter value has no connection with the sample theorem, as is usually supposed [2], and is explained by maximum possible rate of positive and negative pulse change. For low (high) X high (low) frequency of quantization noise is more typical. If absolute value of X is distributed uniformly between 0 and 1, then the constant spectral density of noise applied to the comparator output can be supposed (model of the modulator described in section 1 is assumed, but value of noise evaluation is found by other way). The output spectral density of the modulator is:

$$S_{OUT} = S_{COMP}(1+(\frac{\eta}{\omega\tau})^2)^{-1} \quad (9)$$

where S_{COMP} and η are the spectral density and equivalent gain of the comparator correspondingly, τ - time constant of an integrator. The variance of this signal in frequency range from zero to any frequency f is:

$$\sigma^2(f) = S_{COMP}2f(1-\frac{\eta}{\pi\tau 2f}arctg\frac{\pi\tau 2f}{\eta}) \quad (10)$$

If $f = 0.5f_S$, then (10) must be equal to (4). From this condition S_{COMP} can be found as:

$$S_{COMP} = \frac{1-X^2}{f_S(1-\frac{\eta}{\pi\tau f_S}arctg\frac{\pi\tau f_S}{\eta})} \quad (11)$$

It is clear from (11) that the variance of quantization noise for the comparator is more than for the modulator output. This fact is explained by influence of negative feedback at frequency $0.5f_S$. One can find from (10) and (11):

$$\sigma^2(f) = (1-X^2)\frac{2f(1-\frac{\eta}{\pi\tau 2f}arctg\frac{\pi\tau 2f}{\eta})}{f_S(1-\frac{\eta}{\pi\tau f_S}arctg\frac{\pi\tau f_S}{\eta})} \quad (12)$$

If frequency bandwidth $f << f_S$, then (12) is simplified:

$$\sigma^2(f) = (1-X^2)\frac{(\frac{\pi\tau}{\eta})^2(2f)^3}{3f_S(1-\frac{\eta}{\pi\tau f_S}arctg\frac{\pi\tau f_S}{\eta})} \quad (13)$$

The value of equivalent comparator gain can be chosen from different conditions. For example, a time delay of the true modulator at low frequency must be the same as for linear model. The delay of the modulator for different signal is found within interval from zero till f_S^{-1}. Average value of the delay is 0.5 f_S^{-1}. To realize this delay at low frequency, the equivalent gain of the comparator must be equal to $\eta = 2\tau f_S$. Then (12) and (13) are transformed correspondingly to:

$$\sigma^2(f) = (1-X^2)\frac{(1-\frac{2R}{\pi}arctg\frac{\pi}{2R})}{R(1-\frac{2}{\pi}arctg\frac{\pi}{2})} \quad (12^*)$$

$$\sigma(f) \approx 1.51\sqrt{(1-X^2)/R^3} \quad (13^*)$$

The coefficient R in (12*) and (13*) is the ratio of maximum noise frequency at the output of the modulator $0.5f_S$ to the frequency band f where the quantization noise is considered. The latter frequency can be without any connection with frequency of the input signal. Therefore the coefficient R is better to name as frequency band rejection instead of oversampling ratio accepted now.

If sinusoidal signal with amplitude a is applied to the modulator input, then the variance of quantization noise equals to:

$$\sigma^2(f) = (1-0.5a^2)\frac{2f(1-\frac{\eta}{\pi\tau 2f}arctg\frac{\pi\tau 2f}{\eta})}{f_S(1-\frac{\eta}{\pi\tau f_S}arctg\frac{\pi\tau f_S}{\eta})} \qquad (14)$$

The *SNR* at low frequency for $\eta = 2\tau f_S$ can be found approximately as:

$$SNR \approx 10\lg 0.219 R^3 a^2 /(1-0.5a^2) \qquad (15)$$

It is interesting to say that difference of *SNR* found by new and formal approaches is not so strong as difference of standard deviations.

3. Results of simulation

We used the application program Matlab 6.5 and built-in simulation program Simulink 5 and the standard oversampled Sigma-Delta scheme of A/D converter with the first order modulator shown in Fig. 2. The sampling frequency for this model of ADC is 512·1024 Hz.

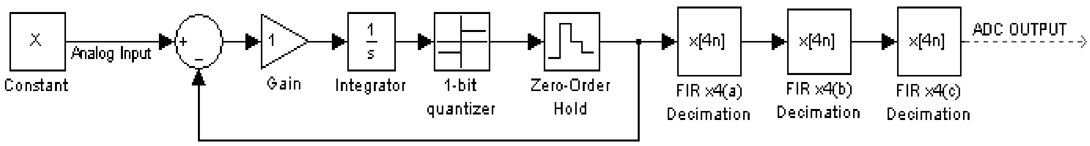

Figure 2. Oversampled Sigma-Delta A/D Converter with first order modulator and approximation loop runs at 512 kHz. Decimation by 64 yields final 8 kHz A/D rate

To check the fundamental equation (4) we applied different randomly chosen constant signals X which taken at the input signal range [0; 1] to the input of the modulator. Only positive range was used because the standard deviation is the even function of X. For each input signal X_i we took sequence of samples from the modulator output with some number of points N. We calculated evaluation of the input signal expectation by using a standard Matlab function "mean", which is equivalent to (5). Standard deviation evaluation was found by using a standard Matlab function "std", which is equivalent to (6). Maximum deflection of calculated results from X and σ given by (4) were close to the errors predicted by (7) and (8). For example, at X = 0.97 and N = 14000, deflection of calculated $\breve{\sigma}$ from σ given by (4) is 0.22 %, deflection predicted by (8) is 0.23 %.
The equation (4) was checked by other method too. For different X_i we built the amplitude spectrum by using Fast Fourier Transform (FFT) with N = 8192 points for each X_i. The standard deviation evaluation $\breve{\sigma}_i$ for X_i was calculated as the sum of root mean squares for all harmonics:

$$\breve{\sigma}_i = \sqrt{\frac{1}{2}\sum_{j=1}^{N/2} U_{m,j}^{\ 2}}\ , \qquad (16)$$

where $U_{m,j} - j^{\text{th}}$ harmonic in spectrum.

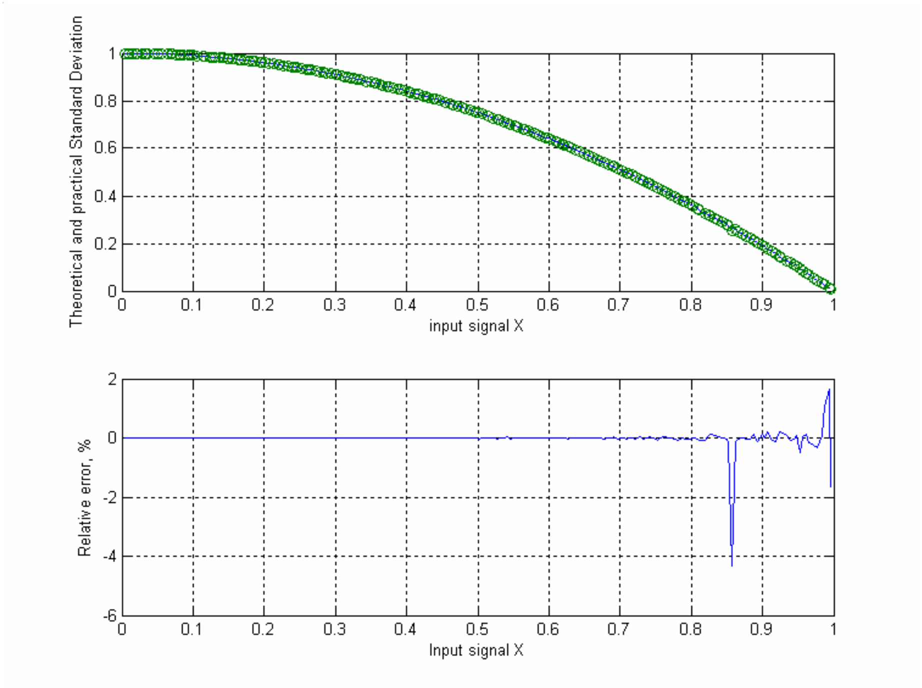

Figure 3. Theoretical and simulation σ vs. input signal and relative error between them

The result is shown in Fig. 3. The maximum difference between (16) and standard deviation calculated by (4) obtains 4.3% at $X = 0.86$. By increasing number of points in FFT, the maximum error within the whole input range [0; 1] decreases with rate approximately -20 dB per decade and for $N = 32768$ it amounts only 0.4%.

All previous simulations correspond to the whole frequency range from 0 to $f_s/2$.

We calculated $\breve{\sigma}_i$ by (16) in lesser frequency range to check (12). We applied different constant signals X_i and built relation between $\breve{\sigma}_i$ and frequency bandwidth. The results for $X = \{0.1; 0.65; 0.95\}$ are shown in Fig. 4. In this figure we can see that the relation for X nearest to 1 ($X = 0.95$) has main jump to low frequency and for small signal ($X = 0.1$) this jump placed to high frequency. The spectrum of signal from modulator output has main harmonic at high (low) frequency for low (high) input signals as was predicted in section 2.

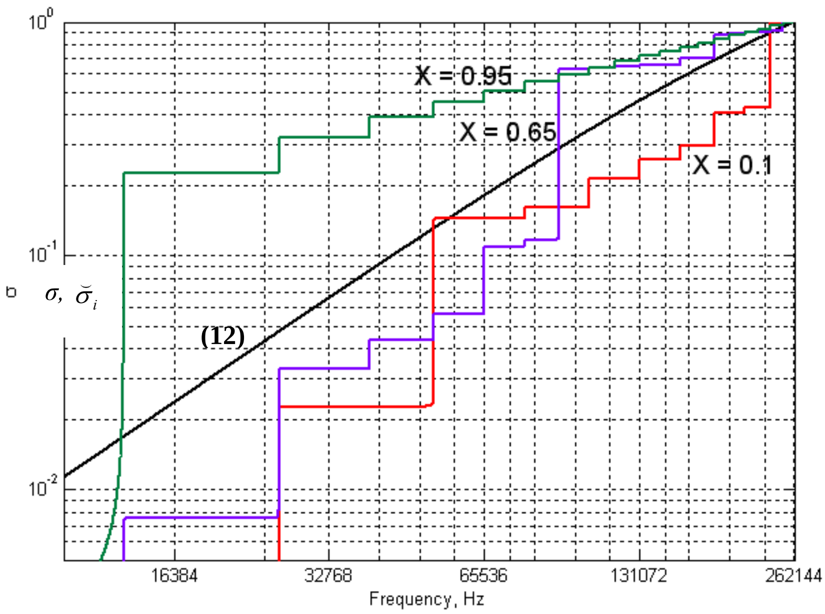

Figure 4. Theoretical σ by (12) and practical $\breve{\sigma}_i$ vs. frequency bandwidth for some constant input signals X_i

For some constant input signals X_i, the difference between (12) and simulation results can be huge (see Fig. 4), but the average value of many realizations for different X_i, uniformly distributed within the input signal range [0; 1], seems to be close to (12). The results of such simulations are represented in Table 1 for different number of points X_i (changed from 10 to 500). From the output of modulator were taken N = 8192 points for every X_i and the average values of standard deviation evaluation was calculated for all X and given frequency.

Number of points X_i	10	20	50	100	200	500
η_{eq}	20207304	1358984	1173034.8	1170978.8	1148612.8	1153304.6
$\eta_{eq}/(2\cdot\tau\cdot f_s)$	19.2712	1.2960	1.1187	1.1167	1.0954	1.0999
Average relative error, %	-7.96892	-1.28861	-0.13863	-0.349928	0.046945	0.04313
Root of relative errors average squares sum, %	24.2766	9.52540	4.52115	5.00958	1.98381	1.43173

Table 1. Relation between parameters and number of points

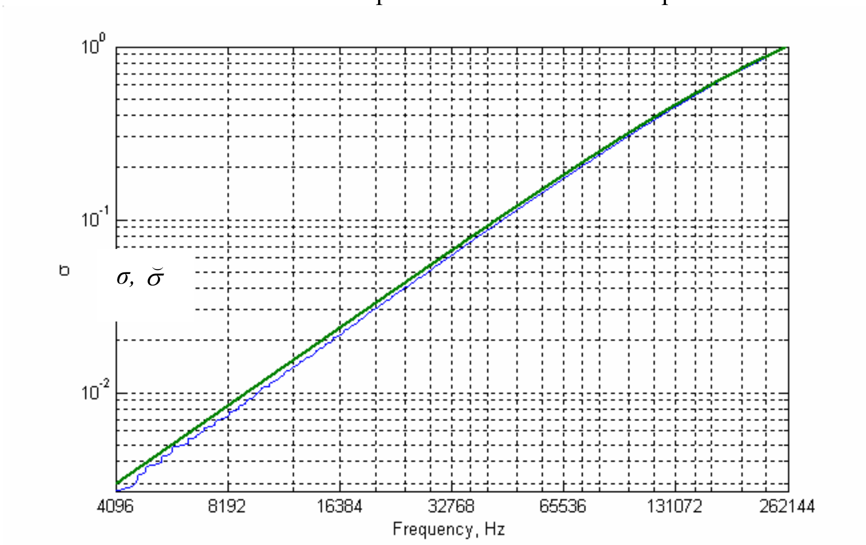

Figure 5. Theoretical σ (thick line) by (12) and average standard deviation evaluation $\breve{\sigma}$ for different X (thin line) vs. frequency bandwidth

The equivalent gain of the comparator was chosen $\eta = 2\tau f_S$ in section 2. Now we will define such value η_{eq} of this coefficient that gives the minimum root of relative errors average squares sum. The relative errors were found as deflection of the average values of standard deviation evaluation and (12). From Table 1 some conclusions can be made. If the number of points X_i increases then the value of equivalent comparator gain η_{eq} is going to the value $\eta = 2\tau f_S$ predicted in section 2. The average relative error (%) and the root of relative errors average squares sum (%) are decreasing.

Theoretical and simulation results for 500 points are shown in Fig. 5.

In Fig. 6 the relations between $\breve{\sigma}_i$ and frequency bandwidth for sinusoidal signals with amplitudes X = {0.1; 0.65; 0.95} are shown. The relation for amplitude nearest to 1 (X = 0.95) has main jump in low frequency and for small amplitudes (X = 0.1) this jump placed in high frequency. The spectrum of signal from modulator output has main harmonic at high (low) frequency for low (high) amplitudes as was predicted in section 2 and similar to results for constant input signals. Due to this fact, *SNR* for higher (lower) amplitudes will be less (more) than predicted by (15).

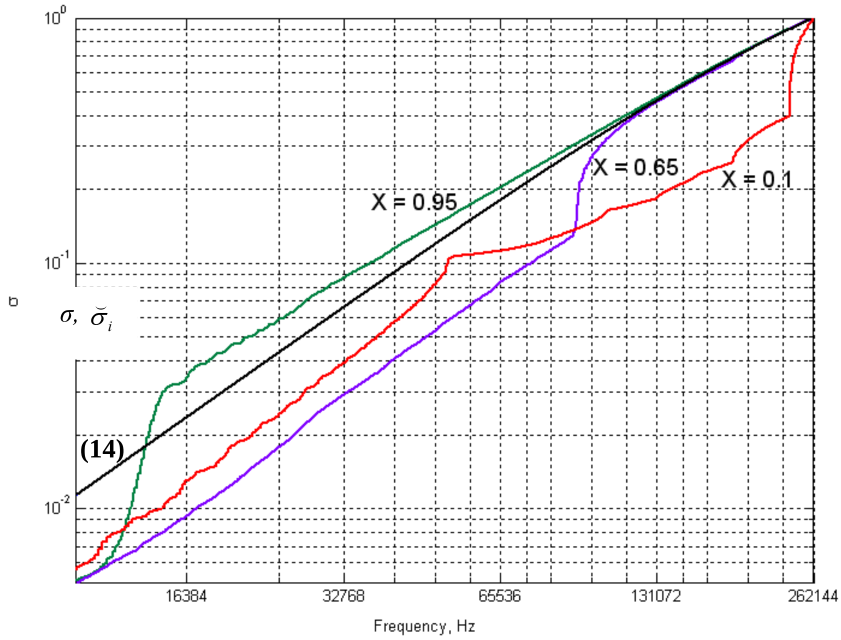

Figure 6. Theoretical σ (14) and practical $\breve{\sigma}_i$ vs. frequency bandwidth for a few of input sinusoidal signal with amplitude X_i and frequency 100 Hz

To check the expectable delay of the modulator, the sinusoidal signals with 100 Hz, 1 kHz and 10 kHz frequencies f_{in} were applied to the input. The phase delay φ between input and output signals was measured by using FFT. The time delay of the modulator $t_m = \varphi/2\cdot\pi\cdot f_{in}$ at frequency f_{in} distinguished from theoretical (section 2) $t_m = T_s/2 = 1/2f_s$ less than 0.02 ppm.

4. Conclusions

The foundation of new approach to the theory of sigma-delta is application of the discrete two-level distribution law instead of the uniform distribution law accepted before. Due to this new approach, the variance of the quantization noise is found to have parabolic dependence on input signal (4) instead of additive error as was supposed before. New equations for variance within frequency range are found. Deflections of these equations received by simulation are decreasing to any negligible level if number of calculation points uniformly distributed within the input signal range [0; 1] is large enough. For high (low) *X* low (high) frequency of quantization noise is more typical and *SNR* will be less (more).

References

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