

Statistical properties of an ML estimator for static ADC testing

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Abstract—A maximum likelihood estimator is derived for the problem of measuring the code transition levels of an ADC. The proposed method is intended to characterize the ADC in the static regime, using only constant test signals, except for a small amount of additive noise. The measurement data are employed in a nearly optimal manner, due to the statistical properties of the maximum likelihood estimator, which are thoroughly examined. The reported analysis allows the design of the test under a given uncertainty constraint.

I. Introduction

Static testing of A/D converters (ADC) is usually considered less important than dynamic one, since many error phenomena do not appear with constant input signals, while arise and become dominant when digitizing dynamic signals [1]. In some cases, however, testing with static signal is mandatory, like for example when assessing the linearity of a digital instrument for static measurements, like digital multimeters (DMMs).

In a previous paper [2], the authors have described a new test method for static ADCs, designed with the aim of overcoming the drawbacks of the existing ones. The advantages of the new method can be outlined as follows.

- Contrary to the procedure in IEEE Standard 1057/94 [3], the method does not require the ability to produce many constant signals for each quantization bin. It is sufficient just one constant signal per bin (and therefore a signal source with much lower resolution), since the method make use of additional information obtainable via dithering.
- Contrary to the procedure in IEEE Standard 1241/00 [4], [5], the method does not require a perfectly timed ADC-DAC feedback loop, and it can be executed even by off-line processing of previously (and/or remotely) acquired ADC data. It is therefore suitable to test stand-alone digital instruments and for inter-laboratory work.
- Like the histogram test based on small triangular waves [6], the new method requires a much shorter test time with respect to the IEEE 1057/94 method. But, contrary to the method in [6], which uses signals with a dynamic of 10-50 LSBs, in the new test the DUT is stimulated by a signal with a dynamic of 1-2 LSBs, due to a small Gaussian noise superimposed to the reference input voltage. This makes the method also much different with respect to the one described in [7], which uses a large Gaussian signal. In general, the proposed method is more trustworthy and better suited for testing instruments like DMM, i.e. instruments designed for constant signals, and without a precise triggering and sampling mechanism, which is needed to acquire correctly a triangular wave.

In [2] the authors have shown, by a preliminary theoretical analysis and by practical experiments, that the proposed method is able to obtain good measurements with very little testing time and cheap testing instrumentation. A thorough uncertainty analysis was not available at that time. This paper, therefore, is aimed at filling this gap.

In Section II the mathematical foundations of the method, based on the construction of a maximum likelihood (ML) estimator for the ADC threshold levels, are illustrated. In Section III the solution of the ML problem is discussed. In Section IV a simpler version of the method, based on the transformation of the ML in a least squares (LS) estimation, is illustrated. In Section V the statistical properties of the ML estimator and the uncertainty of the method are thoroughly examined, and rules for designing the test are given.

II. Derivation of the ML estimator

In the proposed method, the DUT is used to measure, for N times, each of M constant test signals. The measurand is described by a random matrix $\left[X_i^n \right]$, with $i = 1, 2, \dots, M$ and $n = 1, 2, \dots, N$. The

random variables X_i^n are assumed to be independent and normally distributed as $N(x_i, \sigma^2)$. The M input signals are then characterized by their known means $\mathbf{x} = [x_1, x_2, \dots, x_M]$, and by the standard deviation σ of additive white Gaussian noise.

Let t be a code transition level to be estimated. The output codes of the DUT discriminate between the two events $X_i^n \leq t$ and $X_i^n > t$. It is then possible to define a new random matrix $[B_i^n]$, where $B_i^n = 1$ if $X_i^n \leq t$ and $B_i^n = 0$ if $X_i^n > t$, and a vector of relative frequencies $\mathbf{F} = [F_1, F_2, \dots, F_M]$, where $F_i = \frac{1}{N} \sum_{n=1}^N B_i^n$. The random variables NF_i are independent, assume values in the set

$\{0, 1, 2, \dots, N\}$, and are binomially distributed as $B\left(N, \Phi\left(\frac{t-x_i}{\sigma}\right)\right)$.

The values of t and σ can be estimated from a vector $\mathbf{f} = [f_1, f_2, \dots, f_M]$ of observed relative frequencies, realization of \mathbf{F} . The probability of observing \mathbf{f} , given \mathbf{x} , t and σ , is

$$p(\mathbf{f}; \mathbf{x}, t, \sigma) = P(\mathbf{F} = \mathbf{f}; \mathbf{x}, t, \sigma) = \prod_{i=1}^M \binom{N}{NF_i} \Phi^{NF_i}\left(\frac{t-x_i}{\sigma}\right) \left[1 - \Phi\left(\frac{t-x_i}{\sigma}\right)\right]^{N(1-f_i)}. \quad (1)$$

Equation (1) is used to calculate \hat{T} and $\hat{\Sigma}$, the ML estimators of the parameters t and σ . They can be expressed as

$$[\hat{T}, \hat{\Sigma}] = \mathbf{g}(\mathbf{F}, \mathbf{x}). \quad (2)$$

Given \mathbf{f} , the estimates are then

$$[\hat{t}, \hat{\sigma}] = \mathbf{g}(\mathbf{f}, \mathbf{x}). \quad (3)$$

The function \mathbf{g} is, by definition of ML estimator,

$$\mathbf{g}(\mathbf{f}, \mathbf{x}) = \arg \max_{t', \sigma'} p(\mathbf{f}; \mathbf{x}, t', \sigma') \quad (4)$$

with the condition $\sigma' > 0$.

After calculating the log-likelihood function, the following equation is obtained, equivalent to Eq. (4):

$$\mathbf{g}(\mathbf{f}, \mathbf{x}) = \arg \max_{t', \sigma'} \sum_{i=1}^M p'\left(f_i, \frac{t'-x_i}{\sigma'}\right), \quad (5)$$

where

$$p'(f, z) = f \log \Phi(z) + (1-f) \log [1 - \Phi(z)] \quad (6)$$

The ML estimator can be statistically characterized by calculating the bias and the covariance matrix:

$$\mathbf{b}(\mathbf{x}, t, \sigma) = E[\hat{T}, \hat{\Sigma}] - [t, \sigma], \quad \mathbf{C}(\mathbf{x}, t, \sigma) = \begin{bmatrix} \text{var}(\hat{T}) & \text{cov}(\hat{T}, \hat{\Sigma}) \\ \text{cov}(\hat{T}, \hat{\Sigma}) & \text{var}(\hat{\Sigma}) \end{bmatrix}, \quad (7)$$

where \hat{T} and $\hat{\Sigma}$ are obtained using Eq. (2) and \mathbf{F} is distributed according to Eq. (1).

III. Solution of the ML equation

Equation (4) can be solved numerically. The existence of solutions in the right-half-plane ($\sigma' > 0, t'$) is discussed in this section.

It can be proved that in the right-half-plane $p(\mathbf{f}; \mathbf{x}, t', \sigma')$ has only one local maximum. This is essentially a consequence of the concavity of the function $p'(f, z)$ in the variable z , and simplifies the search of the solution of Eq. (4).

However, there are cases in which the maximum is reached asymptotically and the estimates diverge. It can be proved that the estimates are finite and lie in the half-right-plane ($\sigma' > 0, t'$) if

$$\text{cov}(\mathbf{x}, \mathbf{f}) = \sum_{i=1}^M (x_i - \bar{x})(f_i - \bar{f}) < 0, \quad (8)$$

where $\bar{x} = \frac{1}{M} \sum_{i=1}^M x_i$ and $\bar{f} = \frac{1}{M} \sum_{i=1}^M f_i$.

The estimates diverge if $\text{cov}(\mathbf{x}, \mathbf{f}) \geq 0$. The case $\text{cov}(\mathbf{x}, \mathbf{f}) = 0$ isn't of practical importance, because of the uncertainty in the measurement of \mathbf{x} .

Moreover, the conditions in which the maximum of $p(\mathbf{f}; \mathbf{x}, t', \sigma')$ is reached for $\sigma' \rightarrow 0^+$ and $t' \rightarrow t_0$ have been discussed and t_0 determined. This happens if the interval V , defined as follows, is not an empty set:

$$V = \bigcap_{i=1}^M V_i \quad (9)$$

where $V_i = [x_i, +\infty[$ if $f_i = 1$, $V_i =]-\infty, x_i]$ if $f_i = 0$, $V_i = [x_i, x_i]$ otherwise. t_0 is contained in V : it is the middle of V if V is finite, and is infinite if V is semi-infinite.

IV. ML and LS estimation

The Least Square (LS) estimation is an alternative to the ML estimation, and will not be analyzed thoroughly in this paper. An LS estimator can be obtained as follows, observing that the terms

$p' \left(f_i, \frac{t' - x_i}{\sigma'} \right)$ in Eq. (5) are maximized for

$$\frac{\hat{t} - x_i}{\hat{\sigma}} = \Phi^{-1}(f_i). \quad (10)$$

Combining Eqs. (10), with $i = 1, 2, \dots, M$, a linear system in the unknowns \hat{t} and $\hat{\sigma}$ is obtained, which can be solved in the Least Square sense.

In the particular case $M = 2$, the ML estimation and the LS one coincide if the terms $\Phi^{-1}(f_i)$ are finite, that is Eq. (3) is equivalent to

$$\begin{bmatrix} 1 & -\tilde{z}_1 \\ 1 & -\tilde{z}_2 \end{bmatrix} \begin{bmatrix} \hat{t} \\ \hat{\sigma} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad (11)$$

where $\tilde{z}_i = \Phi^{-1}(f_i)$. Equation (11) admits finite solutions, with $\hat{\sigma} > 0$, if and only if $\tilde{z}_1 > \tilde{z}_2$.

In the singular cases $\tilde{z}_1 = +\infty$ or $\tilde{z}_2 = -\infty$ (or both), from Eq. (9) it follows that $t_0 = x_2$ if \tilde{z}_2 is finite; $t_0 = x_1$ if \tilde{z}_1 is finite; $t_0 = (x_1 + x_2)/2$ if \tilde{z}_1 and \tilde{z}_2 are both infinite.

V. Analysis of the ML estimator statistical properties

The statistical properties of the ML estimator depends on the parameters \mathbf{x} , t , σ , M , N . The degrees of freedom in the analysis can be reduced observing that, from Eq. (1), $p(\mathbf{f}; a\mathbf{x} + k, at + k, a\sigma) = p(\mathbf{f}; \mathbf{x}, t, \sigma)$ for any k and for any $a > 0$. Moreover from Eq. (5) it follows that, if $[\hat{t}, \hat{\sigma}] = \mathbf{g}(\mathbf{f}, \mathbf{x})$, then

$$[a\hat{t} + k, a\hat{\sigma}] = \mathbf{g}(\mathbf{f}, a\mathbf{x} + k) \quad (12)$$

for any k and for any $a > 0$. Using the two previous results, it is easily demonstrated that

$$\mathbf{b}(a\mathbf{x} + k, at + k, a\sigma) = a\mathbf{b}(\mathbf{x}, t, \sigma) \text{ and } \mathbf{C}(a\mathbf{x} + k, at + k, a\sigma) = a^2 \mathbf{C}(\mathbf{x}, t, \sigma). \quad (13)$$

In this paper \mathbf{x} is assumed to be perfectly known. However, if there was a bias in the measurement of \mathbf{x} , then from Eq. (12) it follows that \hat{t} would be biased of the same amount.

In the following subsections, the Root Mean Square Error (RMSE) of the estimator and the Cramér-Rao lower bound have been calculated. Moreover the design plots are presented which allows to determine the test conditions for a given accuracy.

A. Cramér-Rao lower bound

A general property of ML estimators is the asymptotic unbiasedness and efficiency. The covariance matrix of the estimator, $\mathbf{C}(\mathbf{x}, t, \sigma)$, reaches the Cramér-Rao lower bound (CRLB), $\mathbf{I}^{-1}(\mathbf{x}, t, \sigma)$, in the limit $N \rightarrow \infty$. The Fisher information matrix \mathbf{I} has been calculated, obtaining:

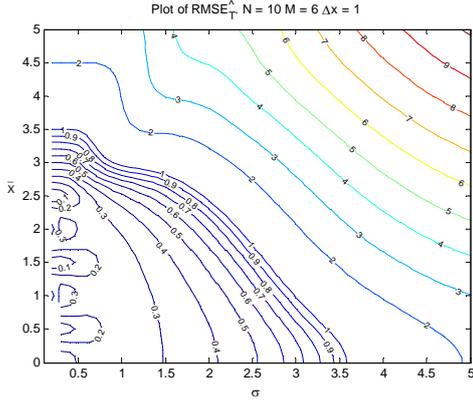


Figure 1. RMSE of the estimator \hat{T} , evaluated for $N = 10$, $M = 6$, $t = 0$ and $\Delta x = 1$.

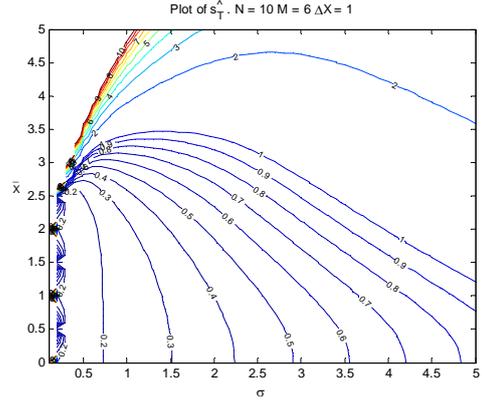


Figure 2. Cramér-Rao lower bound $s_{\hat{t}}$, evaluated for $N = 10$, $M = 6$, $t = 0$ and $\Delta x = 1$.

$$\mathbf{I}(\mathbf{x}, t, \sigma) = \frac{N}{\sigma^2} \sum_{i=1}^M h(z_i) \begin{bmatrix} 1 & -z_i \\ -z_i & z_i^2 \end{bmatrix}, \text{ where } h(z) = \frac{\phi^2(z)}{\Phi(z) \cdot [1 - \Phi(z)]} \text{ and } z_i = (t - x_i) / \sigma.$$

It is noteworthy that \mathbf{I}^{-1} is proportional to σ^2 / N and that $\mathbf{I}^{-1}(a\mathbf{x} + k, at + k, a\sigma) = a^2 \mathbf{I}^{-1}(\mathbf{x}, t, \sigma)$ for any k and for any $a > 0$.

B. RMSE of the ML estimator

The RMSE of the ML estimator and the Cramér-Rao lower bound have been numerically calculated for different values of N and M .

To simplify the analysis of the statistical properties, it is assumed that

$$x_i = (i - 0.5 - M/2)\Delta x + \bar{x}, i = 1, 2, \dots, M, \quad (14)$$

hence the input vector \mathbf{x} is entirely described by M , by the mean of the inputs \bar{x} , and by a fixed mutual distance Δx . Because of properties (13), it is assumed without loss of generality that $\Delta x = 1$ and $t = 0$, or that $\sigma = 1$ and $t = 0$.

Let be $b_{\hat{t}} = [\mathbf{b}]_1$, $\sigma_{\hat{t}}^2 = [\mathbf{C}]_{11}$, $\text{RMSE}_{\hat{t}} = \sqrt{b_{\hat{t}}^2 + \sigma_{\hat{t}}^2}$ and $s_{\hat{t}}^2 = [\mathbf{I}^{-1}]_{11}$.

Figure 1 and Fig. 2 show respectively $\text{RMSE}_{\hat{t}}$ and the Cramér-Rao lower bound $s_{\hat{t}}$, plotted as functions of σ and \bar{x} . $\text{RMSE}_{\hat{t}}$ and $s_{\hat{t}}$ are evaluated in the same conditions: $N = 10$, $M = 6$, $t = 0$ and \mathbf{x} given by Eq. (14), with $\Delta x = 1$. $\text{RMSE}_{\hat{t}}$ and $s_{\hat{t}}$ are similar in those regions in which the dependence on \bar{x} is weak, that is where the range of the inputs is sufficiently large compared to σ and σ is not too small compared to Δx . $\text{RMSE}_{\hat{t}}$ is minimized for $\sigma / \Delta x$ about 0.7, while $s_{\hat{t}}$ is minimized for $\sigma / \Delta x$ about 0.4. These two optimal ratios become closer for $N \rightarrow \infty$.

The bias $b_{\hat{t}}$ is not reported here for the sake of conciseness, however it is noteworthy that, when $\bar{x} = t$, it is $b_{\hat{t}} = 0$ and then $\text{RMSE}_{\hat{t}} = \sigma_{\hat{t}}$. Moreover, if t and \bar{x} are sufficiently close so that the

assumption $t = \bar{x}$ can be made, then the CRLB can be calculated as $s_{\hat{t}}^2(\mathbf{x}, t, \sigma) = \frac{\sigma^2}{N} / \sum_{i=1}^M h(z_i)$. This

result coincides with the CRLB for the estimation of the parameter t when σ is known.

Further considerations about the efficiency of the proposed ML estimator are illustrated at the end of the following subsection.

C. Design of the experiments

From the observation that input values x_i too distant from the threshold t don't improve the estimation, it follows a natural criterion for the choice of M , which consists in calculating the smallest value of M that satisfies the conditions $|z_i| \geq l$, for a given l . Assuming that the input vector is given

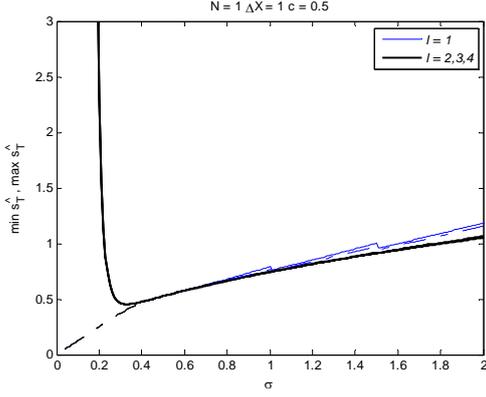


Figure 3. Minimum (dashed line) and maximum (solid line) of the CRLB $s_{\hat{t}}$ as a function of σ , obtained varying \bar{x} according to the constraint (15) with $c = 0.5$. M has been calculated using Eq. (16) for different values of l in the set $\{1,2,3,4\}$.

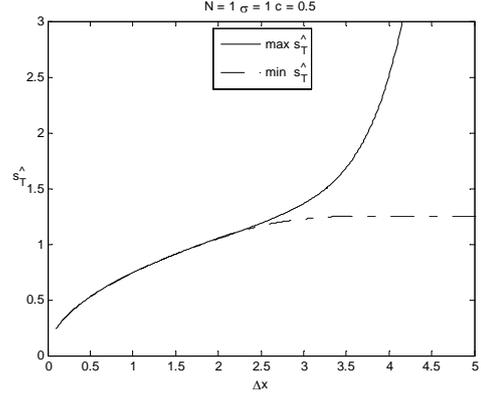


Figure 4. Minimum (dashed line) and maximum (solid line) of the CRLB $s_{\hat{t}}$ as a function of Δx , obtained varying \bar{x} according to the constraint (15) with $c = 0.5$. M has been calculated using Eq. (17).

by (14), M is calculated imposing $z_1 \geq l$ and $z_M \leq -l$, obtaining $M = 1 + \lceil 2l \cdot \sigma / \Delta x + 2|t - \bar{x}| / \Delta x \rceil$. However, the quantity $|t - \bar{x}|$ is unknown and, to calculate the worst case value of M , it is assumed

$$|t - \bar{x}| / \Delta x \leq c, \quad (15)$$

from which it follows that

$$M = 1 + \lceil 2l \cdot \sigma / \Delta x + 2c \rceil. \quad (16)$$

An appropriate value for the parameter l is obtained observing Fig. 3, where $s_{\hat{t}}$ is plotted as a function of σ for different values of l . Being $s_{\hat{t}}$ also a function of \bar{x} , the minimum and the maximum of $s_{\hat{t}}$ are calculated and plotted for different values of \bar{x} , using the constraint (15) with $c = 0.5$. From Fig. 3 it is clear that values of l greater than 2 don't improve further $s_{\hat{t}}$. Moreover, for $l = 2$ and $\sigma / \Delta x \geq 0.4$, $s_{\hat{t}}$ is practically independent on \bar{x} (under the constraint (15)), whereas for $\sigma / \Delta x < 0.4$ the dependence on \bar{x} can not be decreased, even increasing M . Actually, for $\sigma / \Delta x < 0.4$, the number of inputs M can be decreased further, with respect to Eq. (16), without affecting the minimum and the maximum of $s_{\hat{t}}$. In conclusion, it has been found that, for any choice of c , the value of M can be approximately determined as follows:

$$\begin{aligned} M &= 1 + \lceil 2c \rceil, \text{ for } \sigma / \Delta x < 0.3 \\ M &= 2 + \lceil 2c \rceil, \text{ for } 0.3 \leq \sigma / \Delta x < 0.4 \\ M &= 1 + \lceil 4\sigma / \Delta x + 2c \rceil, \text{ for } 0.4 \leq \sigma / \Delta x \end{aligned} \quad (17)$$

Applying Eqs. (17) produces the same results as of Fig. 3 with $l = 2$.

In Fig. 4 $s_{\hat{t}}$ is plotted as a function of Δx . The Fig. 3 and the Fig. 4 are normalized to $N = 1$ and to $\Delta x = 1$ or $\sigma = 1$, respectively. They can be used, with Eqs. (17), to determine the test conditions which satisfy a given uncertainty in the estimates of t , under the hypothesis of efficient estimator and omitting the uncertainty contribution relevant to the measurement of \mathbf{x} .

The hypothesis of efficient estimator has been examined further with the help of Monte Carlo simulations. In Fig. 5 and Fig. 6 each test condition has been simulated 1000 times, giving an estimate of $\text{RMSE}_{\hat{t}}$ which is then compared with the CRLB $s_{\hat{t}}$. Each test condition is defined by the following parameters: N , M , σ , \bar{x} , $\Delta x = 1$, $t = 0$. M is chosen according to Eqs. (17), with $c = 0.5$. For each value of σ , six equally spaced values of \bar{x} have been used, in the interval $[0, c]$. Fig. 5 illustrates the degree of agreement between $\text{RMSE}_{\hat{t}}$ and $s_{\hat{t}}$ for $N = 10$ and for different values of σ . The asymptotical efficiency of the ML estimator is shown in Fig. 6, where different values of N are

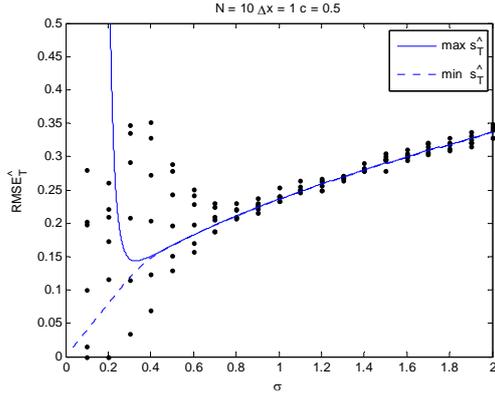


Figure 5. Comparison between $\text{RMSE}_{\hat{T}}$ and $s_{\hat{T}}$ for $N = 10$.

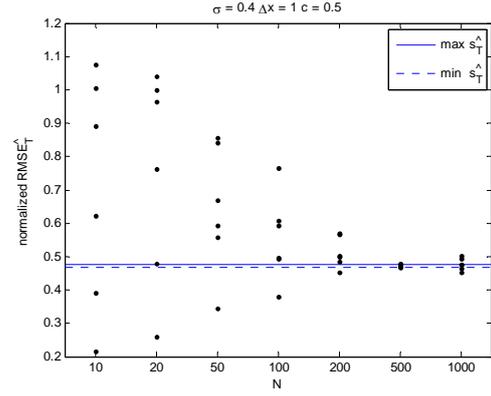


Figure 6. Comparison between the normalized $\text{RMSE}_{\hat{T}}$ and $s_{\hat{T}}$ for $\sigma = 0.4$ and for different values of N .

used for $\sigma = 0.4$. To facilitate comparisons, $\text{RMSE}_{\hat{T}}$ here has been normalized to the case $N = 1$, multiplying it by \sqrt{N} . $\text{RMSE}_{\hat{T}}$ tends towards $s_{\hat{T}}$ for increasing values of N . The apparent inversion of tendency between the results for $N = 500$ and $N = 1000$ is only due to chance, and reveals the accuracy of the Monte Carlo simulations.

VI. Conclusions

The properties of a ML estimator for the purely static measurement of the code transition levels of an ADC have been comprehensively examined. The method is particularly suited for the testing of ADCs in the static regime, because it uses, in an optimal way, only constant test signals dithered by a small amount of noise.

The statistical properties of the ML estimator have been calculated and compared with the Cramér-Rao lower bound. Design plots and formulae have been reported for the determination of the test conditions under a given uncertainty constraint.

In this paper the problem of estimating only two parameters, one threshold and the noise variance, has been addressed. In a similar manner, a ML estimator can be obtained for the estimation of a set of thresholds, and further work involves the statistical characterization of this estimator.

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