Component-wise bounds for MIMO control systems using finite resolution quantizers

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***Abstract*-** This work presents a methodology that guarantees component-wise stability bounds for the states of a disturbed linear Multiple Input Multiple Output (MIMO) feedback control systems, featuring bounded uncertainties and bounded quantization errors both in the measurements and in the controls. The controller synthesis is based on the design of a robust invariant ellipsoidal set, which assumes that all the uncertainties sources introduce upper bounded signals. We show that the component-wise boundedness requirements for the states and controls of the closed loop uncertain system can be expressed as a Linear Matrix Inequality (LMI) problem. Thus, feasibility and optimal controller design can be effectively and advantageously computed solving a convex linear optimization problem. The approach is validated by means of a meaningful numerical example.

I.   Introduction

Quantization in control systems is a well known topic, highlighted by Kalman since sixties [1]. In fact, control systems often use quantization both in the feedback path, where an Analog to Digital Converters (ADC) digitizes sensor readings, and in the control direct path, where a Digital to Analog Converter (DAC) drives the actuators [2][3]. Various approaches can be found in the literature for the estimation of system state ultimate bounds, in presence of disturbances and uncertainty sources, such as the Lyapunov based analysis [4] and the component-wise analysis [5]. These studies discuss boundedness conditions for the various loop signals, and usually rely on worst case analyses, rather than using stochastic models as in [6]. Quantization error introduces a bounded disturbance in the feedback control system. This topic was analyzed in [1][7][8].

Recent developments in network-based feedback control systems have renewed attention on quantization, as it has been recently observed that ADC and DAC quantization, transmission time delays, and limited bandwidth may significantly affect the closed loop performance of such systems [9]-[12]. Static memoryless quantizers are widely used in such applications, due to their low complexity and because they can be combined with dynamic scaling to achieve asymptotic stability [13]. Moreover, logarithmic quantization is often considered, as it requires the coarsest quantization for the quadratic stabilization of LTI systems with quantized feedback [12]. For instance, logarithmic quantizers have been considered for Multiple Input Multiple Output (MIMO) systems, and for output feedback with quadratic or H_∞ performance. Such studies however usually assume an infinite number of quantization levels, which is unpractical. Therefore, more realistic control systems are being studied, based on finite resolution quantizers. For instance, the stability of a Single Input Single Output (SISO) linear control system with finite-resolution logarithmic quantized feedback has been investigated in [14], and a Linear Matrix Inequality (LMI) approach has been used in [15] for the controller design and for the estimation of both the admissible initial states set and the associated attractor set.

In this paper a controller design methodology is introduced, that guarantees component-wise bounds for both states and the control signals of a continuous MIMO system, in presence of bounded modeling uncertainties and bounded quantization error. Memoryless and uniform A/D and D/A converters, have been considered, with finite resolution and Full Scale range. In particular, it is shown that all the design specifications can be expressed as an LMI problem [15]. This is advantageous, as the controller design reduces to the solution of a convex linear optimization problem. The proposed approach is validated by means of a meaningful numerical example.

II.   Problem statement and modeling assumptions

Let us consider a continuous time MIMO system:

$$\dot{x} = Ax + B(u + \delta_{sm}(x)) + \delta_{su}(x)$$
$$y = x + \delta_{sx}(x) \qquad , \qquad (1)$$

where $x \in R^{N_x}$ is the state vector (assumed measurable), $u \in R^{N_u}$ is the control signal, *A* and *B* are known matrices. For the sake of simplicity, let assume that $N_x = N_u$, and *B* is invertible. The considered uncertainty sources

are modeled by the vectors $\delta_{sm}(x)$ $\delta_{su}(x)$ and $\delta_{sx}(x)$, that represent the system matched, unmatched and output uncertainty respectively. The system output y is measured using a quantizer with output v, defined as:

$$v_i = Q_{y_i}(y_i), \qquad i = 0,...,N_x - 1, \tag{2}$$

where $Q_{y_i}(\cdot)$ is the output quantization law of the i-th output channel. The system is controlled by the following:

$$w = Kv \tag{3}$$

linear gain controller, where K is a gain matrix. The obtained control signal w feeds a Digital to Analog Converter (DAC) that provides the actual control signal u:

$$u_i = Q_{u_i}(w_i), \qquad i = 0,...,N_u - 1, \tag{4}$$

where $Q_{u_i}(\cdot)$ describes quantization of the i-th control signal w_i. In this study memoryless quantizers have been considered, so that (2) can be rewritten as:

$$v_i = Q(x_i + \delta_{sx_i}) = x_i + \delta_{sx_i} + \delta_{qx_i} = x_i + \delta_{x_i} \tag{5}$$

where δ_{qx_i} is the quantization error associated to a state measurement, and $\delta_{x_i} = \delta_{sx_i} + \delta_{qx_i}$ is the "overall measurement uncertainty" in the i-th measurement channel. Similarly, for the control channels we have:

$$u_i = Q(kv) = k_i x + k_i \delta_x + \delta_{qu_i}, \tag{6}$$

where k_i is the i-th row of the control matrix K and δ_{qu_i} is the quantization error in the i-th control channel, introduced by a DAC. By using (5) and (6) in (1) we obtain the closed loop dynamics of the MIMO system:

$$\dot{x} = Ax + B(Kx + K\delta_x + \delta_{qu} + \delta_{sm}) + \delta_{su} = (A + BK)x + B(K\delta_x + \delta_u) + \delta_{su}, \tag{7}$$

where $\delta_u = \delta_{qu} + \delta_{sm}$ is the "overall control uncertainty". Let us assume that, for each uncertainty source in (1), the uncertainty upper bound is known. Thus, the following component-wise bounds are assumed:

$$|\delta_{sm_i}| < \Delta_{sm_i}, \qquad i = 0,...,N_u - 1 \qquad x \in D_s$$
$$|\delta_{su_i}| < \Delta_{su_i}, \qquad i = 0,...,N_x - 1 \qquad x \in D_s, \tag{8}$$
$$|\delta_{sx_i}| < \Delta_{sx_i}, \qquad i = 0,...,N_x - 1 \qquad x \in D_s$$

where the Δ_{sm_i}, Δ_{su_i}, and Δ_{x_i} are the known upper bounds, and D_s is a compact set that defines the domain of validity of bounds (8). Let us also assume that quantizers are uniform, bipolar, symmetric and memoryless, each of them with finite resolution and Full Scale range. Without loss of generality, throughout this paper midriser converters will be considered. In particular, the i-th quantizer $Q_{y_i}(y_i)$ does not saturate if $|y_i| \le FS_{y_i}$, where FS_{y_i} is the converter Full Scale range. Moreover, the number of quantization levels is defined as $N_{qy_i} = 2^{b_{y_i}}$, where b_i is the i-th converter resolution, and the corresponding quantizer step is:

$$\Delta_{ADCy_i} = \frac{2FS_{y_i}}{N_{qy_i}} = \frac{2FS_{y_i}}{2^{b_{y_i}}} \qquad i = 0,...,N_x - 1. \tag{9}$$

Provided that $|y_i| \le FS_{y_i}$, quantization error is bounded by:

$$|\delta_{qy_i}| \le \Delta_{qy_i} = \frac{\Delta_{ADCy_i}}{2}, \qquad i = 0,...,N_x - 1. \tag{10}$$

Each $Q_{y_i}(y_i)$ is therefore identified by the number of quantization levels N_{y_i}, the quantizer step Δ_{ADCy_i} and by the maximal quantization error Δ_{qy_i}. Similar definitions hold for the control quantizers $Q_{u_i}(w_i)$, leading to

$$|\delta_{qu_i}| \le \Delta_{qu_i} = \frac{\Delta_{DACu_i}}{2}, \qquad \Delta_{DACu_i} = \frac{2FS_{u_i}}{N_{qu_i}} = \frac{2FS_{u_i}}{2^{b_{u_i}}}, \qquad i = 0,...,N_u - 1. \tag{11}$$

Let us now define the state space region D_x where bound (8) holds and $Q_{y_i}(y_i)$ does not saturate:

$$D_x = \left\{ x \in D_s \,\|\, |y_i| \le FS_{y_i}, i = 0,...,N_x - 1 \right\} = \left\{ x \in D_s \,\|\, |x_i| \le FS_{y_i} - \Delta_{sx_i}, i = 0,...,N_x - 1 \right\}. \tag{12}$$

Note that the bounds of (12) have been translated into corresponding bounds for x, where the system uncertainty δ_{sx_i} appears explicitly. Moreover bounds (8) and (10) may be used to derive a bound for δ_x:

$$\delta_{x_i} \leq \Delta_{sx_i} + \Delta_{qv_i} = \Delta_{x_i}, \qquad i = 0,...,N_x - 1 \qquad x \in D_x. \tag{13}$$

Moreover, in order to prevent the control signals w from saturating the quantizer, we require $|w_i| \leq FS_{u_i}$. The state space region where (8) holds and the saturation of $Q_{ui}(w_i)$ does not occur is defined as:

$$D_u = \left\{ x \in D_s \ \mid \ \left| k_i x + k_i \Delta_x + \Delta_{qu_i} \right| \leq FS_{u_i}, \quad i = 0,...,N_u - 1 \right\} \tag{14}$$

Since within D_u the quantization error is bounded such that $\left| \delta_{qu_i} \right| \leq \Delta_{qu_i}$, then explicit bounds can be derived for the overall control uncertainty δ_u in (7), obtaining:

$$\left| \delta_{u_i} \right| \leq \Delta_{su_i} + \Delta_{qu_i} = \Delta_{ui}, \quad i = 0,...,N_u - 1 \quad x \in D_u. \tag{15}$$

The above bounds quantifies the "size" of the uncertainties in the closed loop dynamics (7) and are essential for the following analysis and for the design phase.

III. Boundedness characterization via invariant sets

This section illustrates a method for characterizing the Uniformly Ultimately Bounded (UUB) regions, in presence of bounded uncertainties, quantization errors and performance requirements. The characterization of the (UUB) region and the derivation of the corresponding controller K will be carried out exploiting the theory of ellipsoidal invariant sets, generated by quadratic Lyapunov functions, as described in subsection III-1. The corresponding bounds on state and control signals are discussed in the following subsection III-2 and III-3.

III-1 Characterization of the UUB region and controller

Let us consider an ellipsoidal set Ω generated by a standard Lyapunov function of the form $V=0.5x^TPx$, with $P=P^T>0$ (symmetric positive definite). Let us also define the set Ω as follows:

$$\Omega = \left\{ x \in \Re^{N_x} \middle| 0.5x^TPx < 1, P = P^T > 0 \right\} \tag{16}$$

Let us assume that Ω is internal to the domain $\overline{D} = D_s \cap D_u \cap D_x$ where all the uncertainties have well defined bounds, that is $\Omega \subset \overline{D}$. Let us also recall the definition of robust positive invariance for set Ω:

Definition: *A set Ω is robustly positively invariant for the uncertain system (7) if, for every initial state $x_0 \square \Omega$, the trajectory of the system remains in Ω for any $t \geq 0$ for any bounded uncertainty $\delta_x(t)\square D_x$, $\delta_u(t)\square D_u$, and $\delta_{su}(t)\square D_{su}$.*

The presented hypotheses allow to derive the following theorem, that gives a sufficient condition for the robust positive invariance of set Ω:

Theorem-1

Consider the system (7) controlled by the linear controller (3) and with bounded uncertainties $\delta_x(t)\square D_x$, $\delta_u(t)\square D_u$, and $\delta_{su}(t)\square D_{su}$. If there exist a scalar $\gamma>0$, a symmetric matrix $P>0$ (with $P = Q^{-1}$), a matrix Y ($Y = PBK$) and three positive diagonal matrices, Ψ_u, Ψ_x and Ψ_{su}, such that:

$$\begin{bmatrix} \begin{matrix} A^TP+Y^T+ \\ PA+Y+\gamma P \end{matrix} & Y & PB & P & 0 \\ Y^T & -\Psi_x & 0 & 0 & 0 \\ (PB)^T & 0 & -\Psi_u & 0 & 0 \\ P^T & 0 & 0 & -\Psi_{su} & 0 \\ 0 & 0 & 0 & 0 & \begin{matrix} \overline{1}\Psi_x\Delta_x^2 + \overline{1}\Psi_u\Delta_u^2 \\ +\overline{1}\Psi_{su}\Delta_{su}^2 - 2\gamma \end{matrix} \end{bmatrix} < 0 \tag{17}$$

then the ellipsoid Ω is robustly invariant with respect to the bounded disturbances $\delta_x(t)\square D_x$, $\delta_u(t)\square D_u$, and $\delta_{su}(t)\square D_{su}$.

Proof: It is sufficient to verify that the time derivative of the Lyapunov function V is ≤ 0 along the boundary of Ω ($\partial \Omega$) for any admissible δ_x, δ_u and δ_{su}. In fact, since by definition of Ω, V is constant along $\partial \Omega$, if the derivative of V is ≤ 0 along $\partial \Omega$ for any admissible δ_x, δ_u and δ_{su} (assuming that $x(0) \in \Omega$), then the trajectory of x cannot escape

the ellipsoid Ω for $t\geq0$. Consequently, Ω is robustly invariant. As a first step the time derivative of V along the system dynamics (7) can be evaluated, obtaining:

$$\dot{V} = 0.5x^T\left((A+BK)^T P + P(A+BK)\right)x + x^T PB(K\delta_x + \delta_u) + x^T P\delta_{su}, \tag{18}$$

Assuming, for the moment, that the set Ω is contained in the region $\overline{D}$, it can be easily verified that the requirements of theorem-1 are satisfied if the condition $\dot{V}$ _ _ is true for each δ_x, δ_u and δ_{su}, such that:

$$0.5x^T Px \geq 1, \tag{19}$$
$$\delta_{x_i}^2 \leq \Delta_{x_i}^2, \qquad i = 0,...,N_x-1 \tag{20}$$
$$\delta_{u_i}^2 \leq \Delta_{u_i}^2, \qquad i = 0,...,N_u-1 \tag{21}$$
$$\delta_{su_i}^2 \leq \Delta_{su_i}^2, \qquad i = 0,...,N_u-1 \tag{22}$$

Notice that (19) defines the boundary $\partial\Omega$ and the region outside Ω, while (20)-(22) define the boundedness constraints (8), (11), and (15) for the uncertainties δ_x, δ_u and δ_{su} respectively.

Condition $\dot{V}$ _ _ with the constraints (19-22) can be transformed in an equivalent single unconstrained LMI using the S-procedure [15]. Applying the S-procedure we have that $\dot{V}$ _ _ and (19-22) are satisfied if and only if there exist coefficients $\gamma\geq0$, $\psi_{x_i}\geq0$ $i=0,...,N_x$-1, $\psi_{u_i}\geq0$, $i=0,...,N_u$-1, $\psi_{su_i}\geq0$, $i=0,...,N_u$-1 such that for any $(x, \delta_x, \delta_u, \delta_{su})$ the following condition holds:

$$0.5x^T\left((A+BK)^T P + P(A+BK)\right)x + x^T PB(K\delta_x + \delta_u) + x^T P\delta_{su} + \gamma\left(0.5x^T Px - 1\right) +$$
$$+ \sum_{i=0}^{N_x-1}\Psi_{x_i}\left(\Delta_{x_i}^2 - \delta_{x_i}^T\delta_{x_i}\right) + \sum_{i=0}^{N_u-1}\Psi_{u_i}\left(\Delta_{u_i}^2 - \delta_{u_i}^T\delta_{u_i}\right) + \sum_{i=0}^{N_x-1}\Psi_{su_i}\left(\Delta_{su_i}^2 - \delta_{su_i}^T\delta_{su_i}\right) \leq 0 \tag{23}$$

By defining the matrixes $\Psi_x = diag\left(\psi_{x_0},...\psi_{x_{N_x-1}}\right)$, $\Psi_u = diag\left(\psi_{u_0},...\psi_{u_{N_u-1}}\right)$, and $\Psi_{su} = diag\left(\psi_{su_1},...,\psi_{su_{Nx-1}}\right)$, (23) can be rewritten as:

$$0.5x^T\left((A+BK)^T P + P(A+BK)\right)x + x^T PBK\delta_x + x^T PB\delta_u + x^T P\delta_{su} + \gamma\left(0.5x^T Px - 1\right) +$$
$$+ \left(\overline{1}\Psi_x\Delta_x^2 - \delta_x^T\Psi_x\delta_x\right) + \left(\overline{1}\Psi_u\Delta_u^2 - \delta_u^T\Psi_u\delta_u\right) + \left(\overline{1}\Psi_{su}\Delta_{su}^2 - \delta_{su}^T\Psi_{su}\delta_{su}\right) \leq 0 \tag{24}$$

where $\overline{1}$ is a row vector of ones having proper length. Expressing (24) in matrix form we finally get:

$$\begin{bmatrix} x \\ \delta_x \\ \delta_u \\ \delta_{su} \end{bmatrix}^T [M] \begin{bmatrix} x \\ \delta_x \\ \delta_u \\ \delta_{su} \end{bmatrix} \leq 0, \qquad M = \begin{bmatrix} \begin{pmatrix} (A+BK)^T P + \\ P(A+BK) + \gamma P \end{pmatrix} & PKB & PB & P & 0 \\ (PKB)^T & -\Psi_x & 0 & 0 & 0 \\ (PB)^T & 0 & -\Psi_u & 0 & 0 \\ P & 0 & 0 & -\Psi_{su} & 0 \\ 0 & 0 & 0 & 0 & \begin{pmatrix} \overline{1}\Psi_x\Delta_x^2 + \overline{1}\Psi_u\Delta_u^2 \\ + \overline{1}\Psi_{su}\Delta_{su}^2 - 2\gamma \end{pmatrix} \end{bmatrix} \tag{25}$$

In order to have the above inequality satisfied for any $(x,\delta_x,\delta_u,\delta_{su},1)$ it is sufficient that:

$$M < 0. \tag{26}$$

Finally, by defining $Y = PBK$ we get the equivalent inequality (17). Analyzing (17) we conclude that, if there exist a scalar $\gamma>0$, matrices $P>0$, Y, $\Psi_x>0$, $\Psi_u>0$ and $\Psi_{su}>0$, then $\dot{V}$ _ _ along $\partial\Omega$, for any admissible δ_x, δ_u and δ_{su}. This implies that any trajectory originating inside Ω will remain in Ω indefinitely, and that Ω is robustly positively invariant for system (15).

In addition we can observe that, if condition (14) holds, than also $\dot{V}$ _ _ holds, not only along $\partial\Omega$ but also in region $\overline{D}$ external to Ω. Defining Ω^* as the largest ellipsoidal set ($0.5x^T Px = k$, where k is a constant) such that $\Omega^* \subset \overline{D}$ (clearly $\Omega \subseteq \Omega^*$), we can easily conclude that Ω is also attractive for any trajectory originating outside Ω that is contained Ω^*.

Note-1: Since the B matrix was assumed invertible, then matrix K can be simply recovered from:

$$K = (PB)^{-1}Y. \tag{27}$$

Note-2: Theorem-1 provides a sufficient condition for Ω to be a robust invariant set under the assumption that $\Omega \subset \overline{D}$ but does not provide information about how to guarantee the containment $\Omega \subset \overline{D}$. Typically, constraints

on the states and on the control signals can be expressed via a set of additional LMIs that enforce the containment of Ω within a suitable region of the state space where all the state and control constraints (requirements) are satisfied. In the following subsections, this problem is addressed.

III-2 State Boundedness requirements

Boundedness for the state are here expressed by the following time domain constraints:

$$|x_i(t)| \le \rho_i, \qquad i = 0,...,N_x - 1, \quad t \ge 0 \tag{28}$$

Where $\rho_i > 0$ are *free design parameters*. Defining vectors $g_i^T = [0,...,1,...0]$ where *i-th* element of g_i is unitary, the constraint (27) are equivalent to $|g_i^T x|^2 \le \rho_i^2$. Thus the admissible errors evolve in the compact polyhedral set:

$$\Pi = \left\{ x \in R^n \,\middle\|\, |g_i^T x|^2 \le \rho_i^2; \, i = 0,...,N_x - 1 \right\} \tag{29}$$

In this study it has been assumed that the bounds in (28) are selected from the designer such that $\Pi \subset D_x$. Such condition can be easily imposed, at design stage, by selecting ρ_i such that $\rho_i \le FS_{yi} - \Delta_{sxi}, i = 0,...,N_x - 1$. In order to fulfill constraints (29) the Ω set must be fully contained in the polyhedron Π. Based on simple geometric considerations, it can be shown [16] that Ω is contained in Π ($\Omega \subset \Pi$) if the following LMI condition is satisfied:

$$\begin{bmatrix} P & g_i \\ g_i^T & 0.5\rho_i^2 \end{bmatrix} \ge 0, \qquad i = 0,...,N_x - 1. \tag{30}$$

Condition (29) will be used in the following sections to express the boundedness requirements $\Omega \subset \Pi \subset D_x$ so that Ω is contained in the domain of validity of the bounds for the system uncertainty and measurement quantization error.

III-3 Control Boundedness requirements

Following an approach similar to that used in previous subsection, it is possible to derive Matrix Inequality conditions similar to (27), that define a domain $\Pi_u \subset D_u$ where the conditions $|k_i x + k_i \delta_x + \delta_{qu_i}| \le FS_{u_i}$, $i = 0,...,N_u - 1$, hold. However, the obtained Matrix Inequality cannot be expressed as a *linear* combination of the *optimization variables* P and Y, derived from the LMI (17) of theorem-1. In particular, by selecting a different set of the optimization variables (ie: P, $Y=P^{-1}K$) it is possible to achieve LMIs expressing the control boundedness requirements, but condition (17) becomes a quadratic-MI (QMI). Applying standard majorations and Matrix manipulations it is indeed possible to transform these QMI in LMI but the price to pay is that these new LMIs conditions are overlay conservative. This led to develop a two-step design procedure, described in the following section, where the bounds on the control amplitudes are considered in the second phase of the design.

IV. Controller design

The proposed design procedure consists of two steps. In the first step a LMI problem is defined, inserting all the constraints except the control amplitude requirements. In this phase a family of feasible solutions is obtained for the controller. In the second step, a controller is searched within the possible family of solutions, satisfying also the control amplitude requirements.

IV-1 Design Phase-1, problem statement

In the first step of the design we search for solutions to the following problem:

Problem-1 - Select a linear state-feedback control u(t)=Kx such that:
- The ellipsoid Ω is robustly positive invariant for dynamics (7) according to theorem-1.
- For $\delta_x=0$, $\delta_u=0$, $\delta_{su}=0$ dynamics (7) is asymptotically stable.
- The state boundedness requirements (27) are fulfilled.

As shown in section III, if a solution to Problem-1 exists, then Ω is a robust invariant set, and the state boundedness requirements $\Omega \subset \Pi$ also hold. Further, if $\delta_x=0$, $\delta_u=0$ and $\delta_{su}=0$, (19) can be used to show that $\dot{V} = 0.5x^T\left((A+BK)^T P + P(A+BK)\right)x \le 0$, proving that K is a stabilizing controller. As anticipated design phase-1 does not consider the control amplitude constraints.

IV-2 Design guidelines for Phase-1

Considering the possible solutions to Problem-1, we are here interested in selecting one of these having a reduced control authority. Since matrix K depends indirectly on the LMI variables it was not possible to have a direct control on the ||K|| during optimization. An indirect control on K authority can be achieved observing that a small

sized Ω typically implies an high gain control matrix while a large Ω requires a reduced control authority. Therefore we are here interested in the computation of the set $\Omega \subset \Pi$ having maximum "size". Since Ω is an ellipsoid, one approach to maximize its size is to minimize the trace of P. The motivation for this is that $n \cdot trace(P)^{-1} \leq trace(P^{-1})$, where $P \in R^{N_x \times N_x}$, and $trace(P^{-1})$ is the sum of the squared semi-axis lengths of the ellipsoid Ω. This design guideline can be formalized as a linear optimization problem having the cost function $J = \sum_i P_{ii}$ and as optimization variables all the free parameters involved in the LMIs of problem-1. Therefore, an optimization problem is considered, that requires to find the following minimum:

$$\min\left(\sum_i P_{ii}\right), \ \gamma > 0, \Psi_u > 0, \Psi_x > 0, P > 0, Y \tag{31}$$

under the conditions (17) and (29). It must be noticed that problem (31) is not convex due to the presence of the product of the optimization variables γ and P in (17). A standard approach to treat this kind of problem is to solve it for each fixed γ in $[0, \infty)$, and then recover the overall minimum, since for each fixed γ, (30) reduces to

$$\min\left(\sum_i P_{ii}\right), \ \Psi_u > 0, \Psi_x > 0, P > 0, Y, \tag{32}$$

a LMI constrained optimization problem, subject to (17) and (30). Problem (31) can be treated numerically, using for example the LMI solver of the MATLAB Robust Control Toolbox [17]. Infeasibility of (32), for any value of γ, implies that at least one constraint involved in (31) cannot be satisfied, and that the proposed approach cannot solve Problem-1 for the established set of uncertainties and performance requirements.

IV-3 Design guidelines for phase-2
If problem-1 is feasible, an interval $\Gamma = [\gamma_{min} \ \gamma_{max}]$ for γ exists, corresponding to a family of controllers K_γ, compatible with the constraints in (32). The objective of phase-2 is thus to select a controller in K_γ, if existing, that satisfies also the control amplitudes requirements. In fact, since the set of solutions to problem-1 guarantees the condition $\Omega \subset \Pi$, upper bounds for the control signal $w_i = k_i x + k_i \delta_x + \delta_{qu_i}$ can be computed assuming that $x \in \Pi$. Under this assumption, for each controller K_γ upper bound for the control amplitudes can be obtained as follows:

$$\max(|w_i|) \leq |k_i|(\rho_i + \Delta_{x_i}) + \Delta_{u_i} = \bar{M}_{u_i}(\gamma), \quad i = 0, ..., N_u - 1 \tag{33}$$

The final step of the design phase-2 is a test over Γ, that verifies the existence of a least a value $\gamma_0 \in \Gamma$, compatible with all the control constraints:

$$\bar{M}_{u_i}(\gamma_0) \leq FS_{u_i}, \ i = 0, ..., N_u - 1 \tag{34}$$

Controller $K(\gamma_o)$, if exists, guarantees that, within Ω, the saturation of the control quantizers does not occur, thus satisfying also the design requirements on the quantized control signals.

V. Design Example
The methodology has been tested by means of simulations, applied to the following 4th order system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = -\begin{bmatrix} -0.56 & -1.03 & -0.5 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \tag{35}$$

$$y = x + \delta_{sx}$$

where the uncertainties δ_{su} and δ_u have been assumed negligible. Notice that, as the system stability matrix A has 3 unstable open loop poles. The uncertainty δ_x has been modeled as a upper bounded white noise with saturations equal to $\pm\Delta_{sx}$; the bounds for δ_x are therefore $\Delta_{sx} = [0.25; 0.25; 0.25]$.

V-1 Specifications and Controller Synthesis
Componentwise UUB specifications (28) for the state have been assumed to be $\rho = [1;1;1]$, resulting in the performance polyhedron $\Pi = \{x \mid |x_i| \leq 1, i = 0, 1, 2\}$. It was assumed that over the domain Π the measurement quantizers do not saturate and that their error magnitude is bounded by $\Delta_{qy} = [0.25; 0.25; 0.25]$. Control amplitudes constraints ($|w_i| \leq FS_{u_i}$) have been assumed to be $FS_u = [25; 25; 25]$, and an upper bound $\Delta_{qu} = [0.25; 0.25; 0.25]$ has been considered for quantization error. By following phase-1, the LMI problem (32) has been solved,
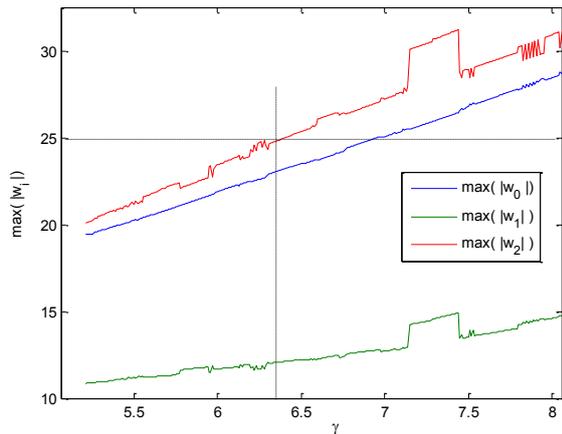
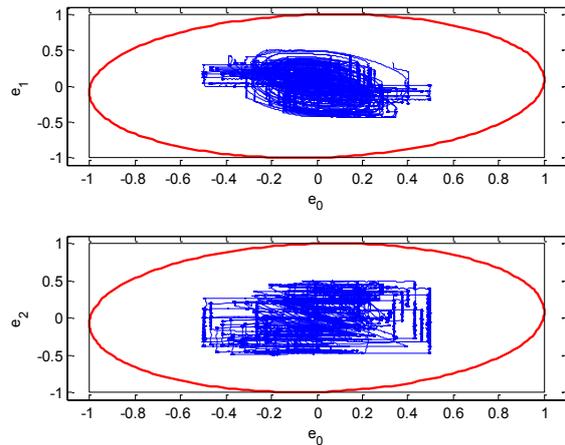Fig.1- Maximum control amplitudes assuming $x \in \Pi$.

Fig.2 - The robust invariant ellipsoidal Ω and the bounding rectangular polyhedron Π.

performing a linear search on γ with step γ=0.01. Feasible solutions have been found, in the interval $\Gamma = [5.25, \infty)$. Then, following phase-2 procedure, a suitable controller in the family K_γ has been selected. Figure-1 shows, for $\gamma \in 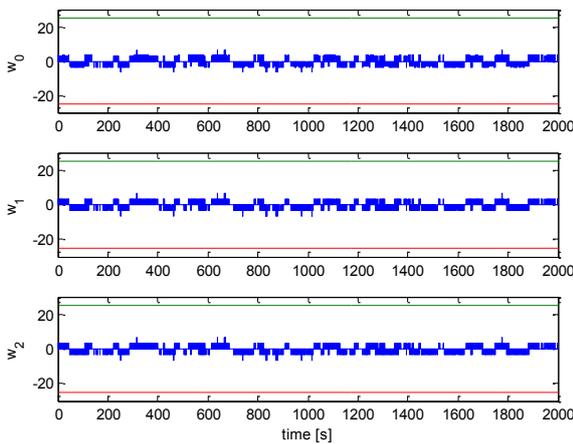\Gamma$, the values of $max(|w_i|)$ i=0,1,2 computed with (33) assuming $x \in \Pi$. It can be observed that for 5.25<γ<6.4 the maximum control signals associated to $K(\gamma)$ fulfill also the control amplitude constraint namely $|w_i| \le FS_{u_i}$, i=0,1,2. For the following closed loop simulations we selected the controller $K(\gamma_0)$ where γ_0=5.3. The other optimization variables, evaluated for γ_0=5.3, were P=[2.03, -0.2, -0.19;-0.2; 2.05, 0.29;-0.19; 0.29, 2.05], Y=[-15.57, 0.05, 0.07; 0.05, -11.2, 0.01; 008, -09, 11.4], Ψ_x=diag[15.55, 11.24, 11.48]; Ψ_u=diag[17.18, 13.49, 15,32]; Ψ_{su}=(5.94, 6.70, 3.78) which resulted in the optimal control matrix: K=inv(PB)Y=[-0.0358, -6.3838, 6.5157; -0.6288, 0.7543, -5.7458;-7.1211, 5.1497, -1.2336]. The three eigenvalues of matrix A+BK are in s=-6.69 and s=-6.69±0.71i. Figure 2 shows planar projections of the robust invariant subset Ω ($0.5x^TPx$=1) generated by $K(\gamma_0)$ and the associated Π

Fig.3 - Quantized control and control amplitude constraints

set. It can observed that, since $\Omega \subset \Pi$, the boundedness constraints are fulfilled, and the uncertainty contribution δ_x and δ_u do not cause the x trajectory to leave Ω. The state trajectory shown in Figure 1 were obtained in a 2000s simulation, that starts from random conditions internal to Ω. Finally, figure 3 shows the quantized controls and the control constraints $|w_i| \le FS_{u_i}$ in the same simulation. It can be observed that also the control signals amplitudes satisfy the design requirements.

VI.   Conclusions

The paper proposes an analysis and controller design methodology for feedback linear MIMO systems affected by model uncertainty, quantized measurement, and both control and component-wise amplitude constraints on these signals. The design process is easy to apply an it is based on a two phase procedure. In the first phase a linear optimization program with LMI constraints is solved to verify the existence of a family of linear controllers that guarantees the component-wise amplitude constraints for the state variables. In the second phase, within this family, it is selected one controller (if it exists) that satisfies also the control amplitudes requirements. A significant simulation study was used to illustrate the design steps as well as the achievable results. Current efforts are directed towards the definition of a single step design methodology that allows also a rectangular B control matrix.

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