Eigenvalue and eigenmode analysis for eddy current sensor model and conducting material

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Abstract- The paper deals with eigenvalue and eigenmode analysis of a model of eddy current sensor over the testing object. This object is made of conducting materials without and with discontinuities. The mathematical model is defined by a Helmholtz equation, and it is the fundamental model to eigenmode analysis. It can be classified as a quasi-static problem in electromagnetics. With finite element method (FEM) the eigenvalues of systems are calculated and the distribution of magnetic flux density (mode) for the eigenvalue are presented.

I. Introduction

Eddy current methods are fast and effective for non-destructive conductivity measurement of materials such as metals, metal alloys and semiconductors. They can be also applied in thickness of walls measurements. The methods are important and oft used for detecting and sizing most of the flaws in conducting materials. Small initial cracks on material surface which can't be detected with ultrasonic testing, could be mostly evaluated with eddy current testing. Eddy current sensor (transducer) can have different geometrical construction. Typically, they have one inductance coil for exciting the electromagnetic field (E) and one or more measurement coils (M) [1-3]. In Figure 1 there are shown two various types of eddy current sensors. The absolute sensor (a) will be used for testing objects with big dimensions. Sensors for testing pipes or wires have usually a construction embracing them as shown in Figure1b.

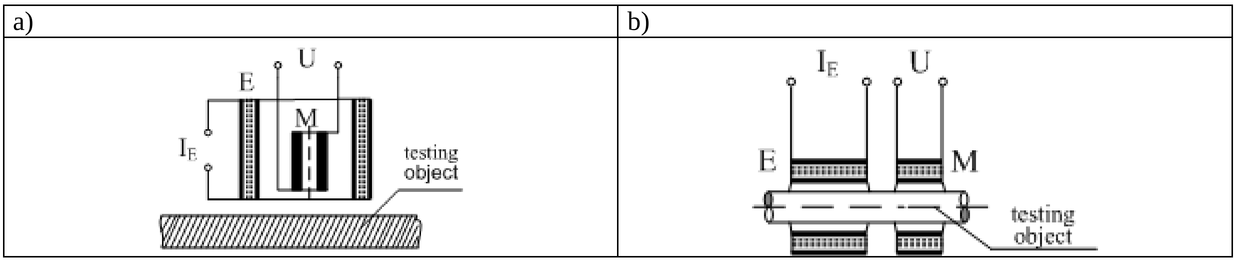


Figure 1. Exemplary schematic diagrams of eddy current sensors, a) surface absolute probes, b) probes with encircling coils, E – exciting coil, M– output coil

In the time harmonic analysis electrical current in the circular exciting coil can be put as $I_E = I \exp(j\omega t)$.
The amplitude and phase of the eddy current output signal in measurement coil depends on several parameters e.g.: conductivity σ and relative magnetic permeability μ_r of materials of the testing object, electrical permittivity ε of sensor enviroment (air), frequency f and current of the excitation coil I_E, distance h between probe and a specimen, temperature T, material errors such as discontinuity or non-homogeneity [4, 5].

II. Mathematical model of a sensor with testing material

The Maxwell's equations for eddy current sensor and conducting material have form

$$\nabla \times \vec{H}(x,y,z,t) = \vec{j}_e(x,y,z,t) + \frac{\partial \vec{D}(x,y,z,t)}{\partial t} \tag{1}$$

$$\nabla \times \vec{E}(x,y,z,t) = -\frac{\partial \vec{B}(x,y,z,t)}{\partial t} \tag{2}$$

$$\nabla \cdot \vec{B}(x,y,z,t) = 0 \tag{3}$$

$$\nabla \cdot \vec{D}(x,y,z,t) = \rho \tag{4}$$

$$\vec{D} = \varepsilon_0 \varepsilon_r \vec{E} + \vec{P} \tag{5}$$

$$\vec{B} = \mu_0 \mu_r \left(\vec{H} + \vec{M}\right) \tag{6}$$

where $\vec{H}$ - magnetic field intensity, $\vec{j}_e$ - current density in a exciting coil, $\vec{D}$ - electric displacement (electric flux density), $\vec{E}$ - electric field intensity, $\vec{B}$ - magnetic flux density, $\vec{P}$ - electric polarization vector, $\vec{M}$ - magnetization vector, ε_0 - permittivity of vacuum, ε_r - relative permittivity of material, μ_0 - permeability of vacuum, μ_r - relative permeability of material, ρ - electric charge density [6, 7].
In case of axial symmetry in cylindrical coordinates for sinusoidal excitation current, the eddy current sensor model can be described with the Helmholtz equation for the magnetic potential vector $\vec{A}$

$$\nabla^2 A_\varphi + k^2 A_\varphi = -\mu_0 \mu_r j_{e\varphi} \tag{7}$$

where

$$k^2 = \omega^2 \mu\varepsilon - \mathrm{j}\omega\mu\,\sigma, \tag{8}$$

$$\mu = \mu_0 \mu_r, \quad \varepsilon = \varepsilon_0 \varepsilon_r, \tag{9}$$

and $j_{e\varphi}$ azimuthal coordinate of current density in the exciting coil, A_φ azimuthal coordinate of magnetic potential vector. For insulator (exemplary air) from equation (8) holds $k^2 = \omega^2\mu\varepsilon$ and for good conductors $k^2 = -\mathrm{j}\omega\mu\,\sigma$. Equation (7) was the basic equation applied to calculate the electromagnetic field around eddy current sensor with Finite Element Method (FEM).

III. FEM simulation multi - coil transducer model

An axial symmetry Comsol model of multi - coil eddy current sensor over testing materials is shown in Figure 2. It consists of one exciting coil E and seven measure coils $M_1 \div M_7$ of 1 mm wire diameters. For numerical simulations, the tested material (iron) is taken of conductivity σ = 10 MS/m and relative permeability μ_r = 5. The sensor coils are made of copper of conductivity $58\cdot10^6$ S/m and μ_r = 1. The flaw is modeled as ring discontinuities (material: air) with a rectangular cross section of height *h* and width 0,5 mm lying in distance *d* under the surface. The thickness of tested specimen (metal plate) is 2 cm.

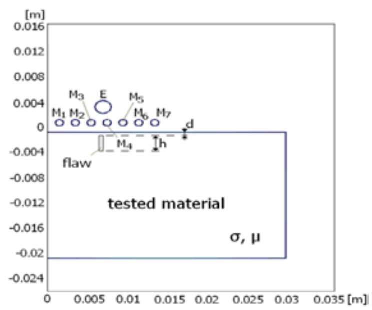
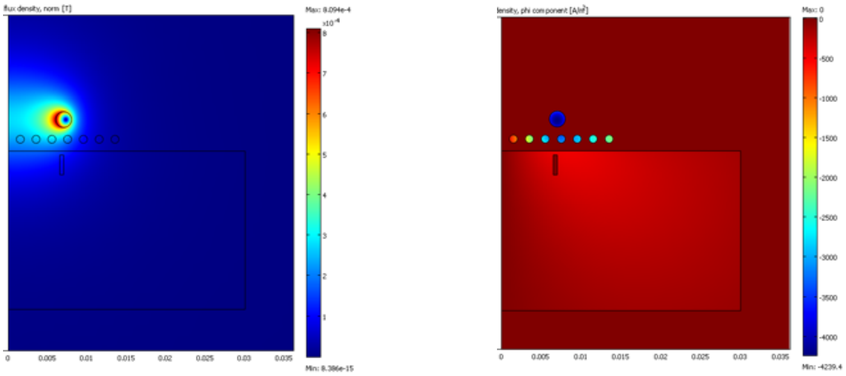


Figure 2. Axial symmetry Comsol model of multi-coil eddy current sensor over the plate of conducting material with rectangular cross section flaw, E – exciting coil, $M_1 \div M_7$ - measure coils

FEM evaluations of distribution of electromagnetic fields for conducting material, have been depicted in Figure 3. For calculations the current density in the excitation coil E was equal j_e = 10^6 A/m^2. There are a distribution absolute value of magnetic flux density *B* norm and eddy current density j_φ.



a) magnetic flux density *B* norm   b) eddy current density j_φ

Figure 3. Magnetic flux density (a) and eddy current density (b) around the sensor model over material with flaw at frequency f = 100 Hz; parameter of material: $\sigma = 10^7$ S/m, $\mu_r = 1$; flaw cross section dimensions - 2,5mm $\times$ 0,5mm

IV. Eigenvalue problem for multi - coil transducer model

The model of eddy current sensor together with testing object made of conducting material can be described with partial differential equation for magnetic vector potential $\vec{A}$ [6, 7]

$$\sigma \frac{\partial \vec{A}}{\partial t} + \nabla \times \left(\mu_0^{-1} \mu_r^{-1} \nabla \times \vec{A} \right) = \vec{j}_e \ , \qquad (10)$$

where $\vec{j}_e$ exciting current density. For axial symmetry model in cylindrical coordinates and with non-zero azimuthal exciting current density vector $j_{e\varphi}$, holds equation

$$\sigma \frac{\partial A_\varphi}{\partial t} + \nabla \times \left(\mu_0^{-1} \mu_r^{-1} \nabla \times A_\varphi \right) = j_{e\varphi} \ . \qquad (11)$$

A homogeneous equation can be written

$$\sigma \frac{\partial A_\varphi}{\partial t} + \nabla \times \left(\mu_0^{-1} \mu_r^{-1} \nabla \times A_\varphi \right) = 0 \ . \qquad (12)$$

When we put

$$\frac{\partial A_\varphi}{\partial t} = -\lambda A_\varphi , \qquad (13)$$

to Eq. (12) we became the form

$$\nabla \times \left(\mu_0^{-1} \mu_r^{-1} \nabla \times A_\varphi \right) = \lambda \sigma A_\varphi , \qquad (14)$$

where λ is element of eigenvalue set of equation. Each eigenvalue corresponds its eigenfunction $\varphi_i(\lambda_i)$. For any coordinate system we can apply a substitution of $\frac{\partial \vec{A}}{\partial t} = -\lambda \vec{A}$ to a vector equation (10).

Deriving of eigenvalues depends on finding such values of λ_i, for whose there exist nontrivial solutions of Eq. (14) with respect of boundary conditions, resulting in evaluation of set of eigenvalues and set of corresponding them eigenfunctions. For homogenous isotropic materials μ_r and σ are scalars, so in Eq. (14) they can be extracted before del operator

$$\sigma^{-1}\mu_0^{-1}\mu_r^{-1}\nabla\times\left(\nabla\times A_\varphi\right)=\lambda A_\varphi\,. \qquad (15)$$

The obtained solution is in form of a sum of products of a function dependent on spatial coordinates and a time function (Ritz series) [7, 8]

$$\vec{A}(x,y,z,t)=\sum_{i=1}^{\infty}\vec{A}_i(x,y,z)\exp(-\lambda_i t)=\sum_{i=1}^{\infty}\varphi_i(\vec{r})\exp(-\lambda_i t), \qquad (16)$$

where $\varphi_i(\lambda_i)$ are eigenfunctions (another name eigenmodes) of magnetic potential vector described with equation (15). For this equation, eigenvalues λ_i are reverse of relaxations time τ_i

$$\lambda_i=\frac{1}{\tau_i}\,. \qquad (17)$$

Eigenvalues can be interpret as inverse of propagation time constants of losses (decay) of disturbances of electromagnetic field in testing materials. A maximal value of time constant can be calculated as an inverse of $\lambda_{\min}$ - minimal value of eigenvalue set.

V. Eigenvalue analysis results

Eigenfunction $\varphi_i(\lambda_i)$ for magnetic flux density for the least four eigenvalues λ_i (i = 1…4) calculated for eddy current sensor model and testing specimen fabricated from ferromagnetic material (Figure 3) are shown in Figure 4.

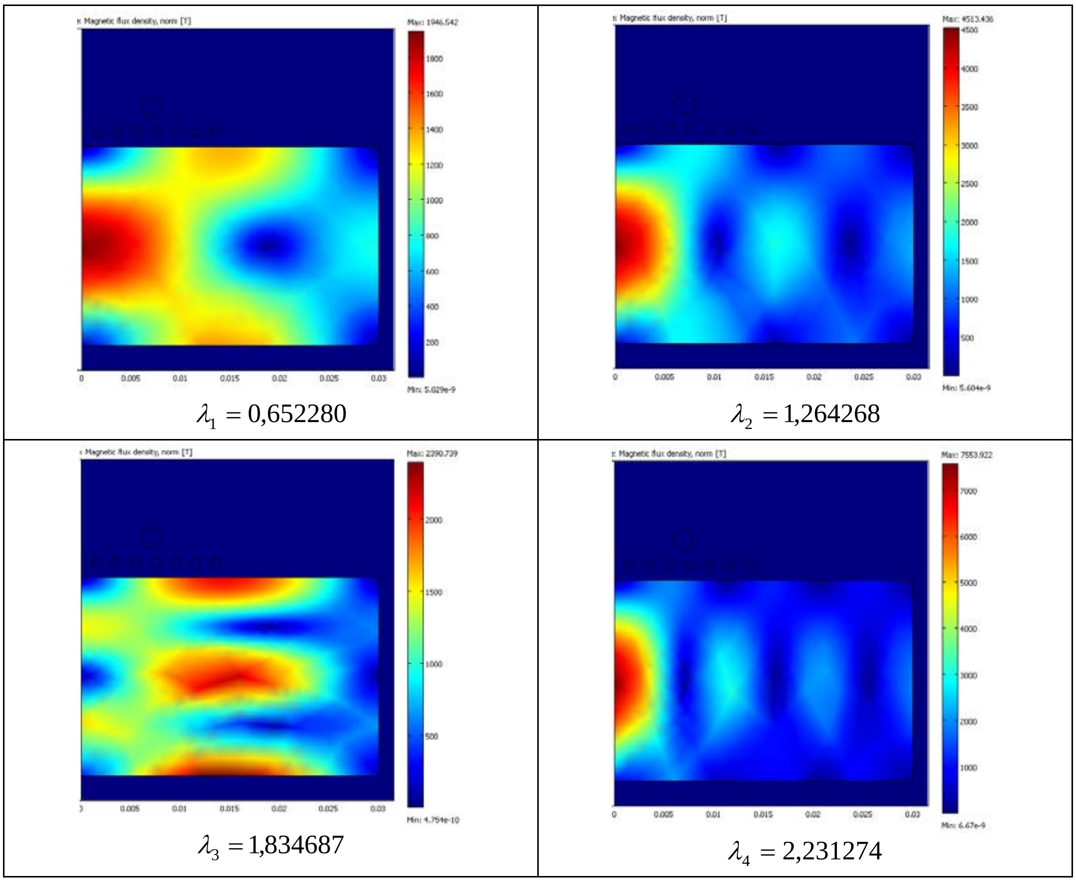


Figure 4. Eigenfunction $\varphi_i(\lambda_i)$ for magnetic flux density (module) and corresponding eigenvalue λ_i (i = 1…4) for multi-coil eddy current sensor model and ferromagnetic material without flaw

The values of eigenvalues are real and positive. The set of eigenvalues has infinite number of elements but practically the crucial properties of the investigated system are approximated by some first of them. The algorithm allows to calculate all values in a given range.

In Figure 5 distribution of the least eigenvalues λ_i (i = 1 ÷ 50), calculated for eddy current sensor model over a specimen of ferromagnetic material without flaw on horizontal axis is presented. This diagram represents an eigenvalue spectrum for investigated system.

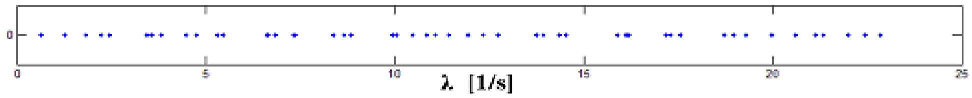


Figure 5. Distribution of eigenvalues λ_i (i = 1 ÷ 50) calculated for eddy current sensor (model from Figure 3) and ferromagnetic material

In Figure 6 is drawn a histogram for eigenvalue set of λ_i (i = 1 ÷ 250) calculated for eddy current sensor model with testing ferromagnetic materials. It is a typical shape of Poisson distribution. The most instants of eigenvalues concentrates at the neighborhood of zero.

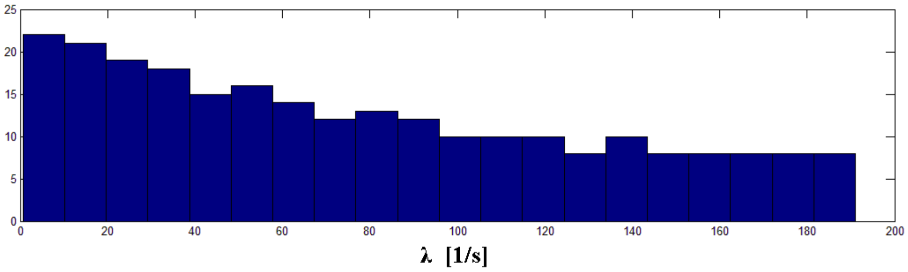


Figure 6. Histogram of eigenvalue set λ_i (i = 1 ÷ 250) for testing system of ferromagnetic material without flaw

A minimal value of eigenvalue set of investigated system depends on conductivity and also on permeability of materials. In Figure 7 the minimal value of eigenvalue set λ_{min} is depicted versus material conductivity in range of (1 ÷ 60) 10^6 S/m – specific for metals and alloys.

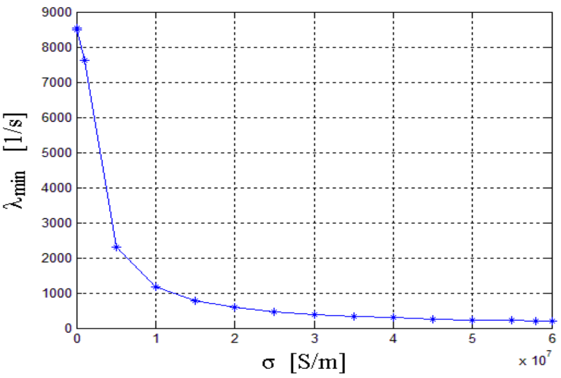


Figure 7. Eigenvalue λ_{min} of system from Figure 3 versus material conductivity

When in material exist flaws such as discontinuities, the eigenvalues of investigated systems are changing. In Figure 8 the results of eigenvalue analysis for eddy current sensor model and ferromagnetic materials without and with flaw are presented. The flaw was modelled as a ring discontinuity with rectangular cross section with height *h* and width *b*, under the surface (in distance to surface 0,5 mm). The electrical and magnetic parameter of flaw have been set as for air.

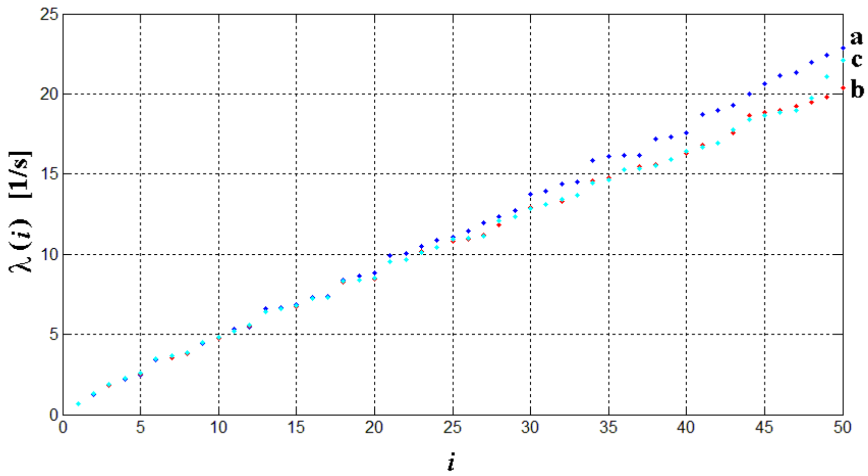


Figure 8. Eigenvalue λ_i (i = 1 ÷ 50) for ferromagnetic materials with- and without flaw
a) material without flaw, b) material with flaw depth h = 2,5 mm and width 0,5 mm, c) material with flaw depth h = 5 mm and width 0,5 mm

VI. Conclusions

The results of eigenvalue analysis for models of electromagnetic systems have shown the coincidence between eigenvalue spectrum of an FEM model for the electromagnetic quasi-static problem, described by the Helmholtz equation, related to testing material parameters and flaw dimensions. Knowledge of relaxation time is inherent for mathematical modeling a some physical, biological or medicine effects. During eigenvalue analysis is possible to describe dynamic properties of mathematical models for this effects.

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