

CALCULATING ASSOCIATED CIRCLE PARAMETERS FOR INTERRUPTED ROUNDNESS PROFILES

¹Dariusz Janecki, ²Stanisław Adamczak, ³Krzysztof Stępień

Faculties of Mechatronics and Machinery Design, Kielce University of Technology,
Al. 1000-lecia P.P. 7, PL-25314 Kielce, Poland

¹Centre for Laser Technologies of Metals, E-mail: djanecki@tu.kielce.pl

²Dean, E-mail: adamczak@tu.kielce.pl

³Chair of Mechanical Technology and Metrology, E-mail: kstepien@tu.kielce.pl

Abstract: Rotary elements constitute a large and important group of machine parts. They are common in various industries, for instance, automotive, steel and paper industries. This is the reason why the problem of accurate measurement of roundness deviation is one of the most important fields of the industrial application of surface metrology. According to the requirements of new Geometrical Product Specifications standards, correct determination of the roundness parameters requires the calculation of the location of the centre and the radius of the associated circle. Usually these values are calculated on the basis of the linearized model describing the value of the measured profile as a function of an angle. The paper presents an iterative algorithm allowing for the calculation of associated circle parameters by the least squares method. It also describes the results of the computer simulation for the non-linear model. The presented algorithm can also be applied in measurements of interrupted roundness profiles when the eccentricity is large in relation to the workpiece diameter.

Key words: Roundness, associated circle, interrupted profile.

1. INTRODUCTION

Rotary elements, common in industries such as automotive, power or papermaking, to name just a few, are regarded as a vital group of machine parts. Accurate roundness deviation evaluation is a very important surface metrology application. The usual guideline is that prior to roundness measurement with a rotary spindle or table, the measured object should be centered so that its axis coincides with the axis of rotation (Whitehouse, 1994).

This enables the measurement at high measuring amplifier gain, which decreases both noise level and errors resulting from the signal quantization. The centering procedure, however, is a tedious process demanding operator's high manual skills, especially when measuring discontinuous profiles. By interrupted profile we understand a roundness profile that consists of a few (or one) separate profile sections. Significant non-concentricity of the measured profile can also occur when measuring concentricity of one profile with respect to the other if both profiles are clearly non-concentric.

Advances in electronics have made it possible to design measuring amplifiers with the high measuring resolution and low noise. In such a case the centering of the object is unnecessary (Rudziński *et al.*, 1995). However, when the centre of the object is quite far from the centre of rotation, the non-linearity of the function linking the real profile and the measured profile plays a significant role, which should be taken into account in the analysis of the measurement results.

2. DETERMINING THE ASSOCIATED CIRCLE CENTER FOR CLOSED PROFILES [3]

Let us consider the XY system of coordinates whose origin O is the rotation point of the table or spindle. The real profile in the polar system with the center O is described by $R_m(\varphi)$. O' is any point on the XY plane. Let the profile in the polar system with the center O' be described by $R'(\alpha)$. The function $R'(\alpha)$ depends on the point O' coordinates' position in XY system (Żebrowska-Łucyk, 2000). Let us denote the coordinates of the point O' in the XY system and the polar system as (e_x, e_y) or (e, δ) respectively, where $e = \sqrt{e_x^2 + e_y^2}$, $\delta = \arctan(e_x, e_y)$. The coordinates (e_x, e_y) and (e, δ) will be used exchangeably, as required.

While determining the center of the object with instruments giving relative indications, the absolute value of the object's profile distance from the center of rotation O is usually not known. Therefore

$$R_m = R_a + \Delta R_m, \quad (1)$$

for a certain R_a , and this value is not known exactly (Janecki *et al.*, 2006). In the majority of cases, however, we know the nominal value of object's radius, R_0 . The value can be aliased with the value of the mean radius. The task of determining the center of the associated circle by the least-squares method (further referred to as mean circle) can be thus formulated in the following way: determine

coordinates (e, δ) of the point O' and the mean value of the measured profile R_a so that for the given value of R_0 , the integral

$$J(e, \delta, R_a) = \frac{1}{2} \int_0^{2\pi} (R'(\alpha) - R_0)^2 d\alpha \quad (2)$$

reaches minimum with respect to variables e , δ and R_a . The solution to the formulated task becomes a lot simpler if the assumption that object's roundness deviation is much smaller than the nominal radius R_0 is satisfied. Fig. 1 will help us consider the case.

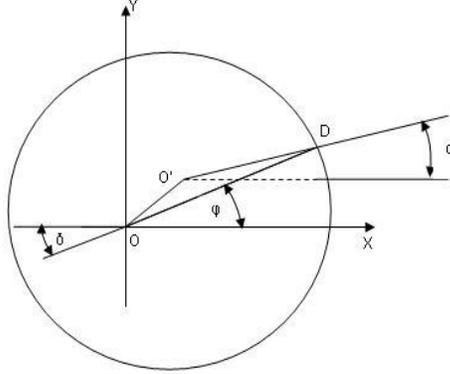


Fig. 1. Mutual position of the center of rotation O and the center of the nominal profile O'

As stated earlier, the coordinates of point O' in the XY system are e_x , e_y , and the distance between O' and the origin of the system, and the angle of inclination between segment $\overline{OO'}$ and X axis are e , δ respectively (thus $e_x = e \cos \delta$, $e_y = e \sin \delta$). Angles between segments \overline{OD} and $\overline{O'D}$ and X axis will be denoted as φ and α respectively. The segments length \overline{OD} and $\overline{O'D}$ dependence on the angle φ will be $R_m(\varphi)$ and $R(\varphi)$; the distance of the segment $\overline{O'D}$ as a function of the angle α will be $R'(\alpha)$. The following relationships result from the fig. 1:

$$R_m(\varphi) = e \cos(\varphi - \delta) + \sqrt{R^2(\varphi) - e^2 \sin^2(\varphi - \delta)}, \quad (3)$$

$$\sin \alpha = \frac{R_m(\varphi) \sin \varphi - e \sin \delta}{R(\varphi)}, \quad (4)$$

$$\cos \alpha = \frac{R_m(\varphi) \cos \varphi - e \cos \delta}{R(\varphi)}. \quad (5)$$

If the roundness deviation is much smaller than the nominal radius, we can assume that

$$\begin{aligned} R_m(\varphi) &= e \cos(\varphi - \delta) + \sqrt{R^2(\varphi) - e^2 \sin^2(\varphi - \delta)} \cong \\ &\cong R_0 (\varepsilon \cos(\varphi - \delta) + \sqrt{1 - \varepsilon^2 \sin^2(\varphi - \delta)}) = \\ &\stackrel{df}{=} R_0 f(\varepsilon, \delta, \varphi), \end{aligned} \quad (6)$$

Where,

$$f(\varepsilon, \delta, \varphi) = \varepsilon \cos(\delta - \varphi) + \sqrt{1 - \varepsilon^2 \sin^2(\varphi - \delta)}, \quad \varepsilon = \frac{e}{R_0} \quad (7)$$

The profile mean value $R_m(\varphi)$ in the range $(0, 2\pi)$ can be then determined from the following dependence

$$\begin{aligned} R_a(e) &= \frac{1}{2\pi} \int_0^{2\pi} R_m(\varphi) d\varphi \cong R_0 \frac{1}{2\pi} \int_0^{2\pi} f(\varepsilon, \delta, \varphi) d\varphi = \\ &= R_0 \frac{1}{2\pi} \int_0^{2\pi} \sqrt{1 - \varepsilon^2 \sin^2(\varphi - \delta)} d\varphi = \frac{2R_0}{\pi} \kappa(\varepsilon) \end{aligned} \quad (8)$$

where, $\kappa(\varepsilon)$ is the complete elliptic integral of the first kind (Spiegel, 1971).

Since the component $\sqrt{1 - \varepsilon^2 \sin^2(\varphi - \delta)}$ of the function $f(\varepsilon, \delta, \varphi)$ is periodic with the period π with respect to the variable φ , we can write

$$\int_0^{2\pi} \sqrt{1 - \varepsilon^2 \sin^2(\varphi - \delta)} \sin \varphi d\varphi = 0$$

and

$$\int_0^{2\pi} \sqrt{1 - \varepsilon^2 \sin^2(\varphi - \delta)} \cos \varphi d\varphi = 0. \quad (9)$$

Thus, using again the assumption that the roundness deviation is much smaller than the nominal radius, we obtain

$$\frac{1}{\pi} \int_0^{2\pi} R_m(\varphi) \cos \varphi d\varphi \cong \frac{1}{\pi} \int_0^{2\pi} e \cos(\delta - \varphi) d\varphi = e_x, \quad (10)$$

$$\frac{1}{\pi} \int_0^{2\pi} R_m(\varphi) \sin \varphi d\varphi \cong \frac{1}{\pi} \int_0^{2\pi} e \sin(\delta - \varphi) d\varphi = e_y. \quad (11)$$

Relationships (10) and (11) show that despite the complex non-linear dependence describing the measured profile, coefficients of the measured profile first harmonic are approximately equal to the mean profile circle center.

The consideration presented above lead to the following way of determining mean circle center and the real profile on the basis of the measured profile ΔR_m .

1. Determine the coordinates of the mean circle from the formulae (10) and (11):

$$e_x \cong \frac{1}{\pi} \int_0^{2\pi} \Delta R_m(\varphi) \cos \varphi d\varphi, \quad e_y \cong \frac{1}{\pi} \int_0^{2\pi} \Delta R_m(\varphi) \sin \varphi d\varphi.$$

2. Calculate the measured profile in absolute values

$$R_m = R_a(e) + \Delta R_m = \frac{2R_0}{\pi} \kappa(e/R_0) + \Delta R_m.$$

3. Calculate the real profile from the relationship (3),

$$\begin{aligned} R(\varphi) &= \sqrt{R_m^2(\varphi) - 2R_m(\varphi)e \cos(\delta - \varphi) + e^2} \\ \alpha(\varphi) &= \arctan(R_m(\varphi) \cos \varphi - e \cos \delta, \\ &R_m(\varphi) \sin \varphi - e \sin \delta). \end{aligned}$$

3. CALCULATING THE CENTRE OF THE ASSOCIATED CIRCLE FOR INTERRUPTED PROFILES

We deal with interrupted profiles when measuring profiles of gear wheels, longitudinal sections of a bearing ring and other. The issue of determining the center of the associated circle for discontinuous profiles was divided into two stages. First the issue was solved for the case when the quotient $\varepsilon = \frac{e}{R_0}$ is small. Then the authors developed the iterative algorithm that helped determine the center of the associated circle for any value $\varepsilon = \frac{e}{R_0}$.

3.1. The Case of a Small Value of the Coefficient ε

Let us denote as Φ a set of angles for which profile ΔR_m is defined. Linearizing the equation (3) in the neighbourhood $e = 0$, we obtain

$$R_m(\varphi) = R(\varphi) + e \cos(\varphi - \delta) = R(\varphi) + e_x \cos \varphi + e_y \sin(\varphi) \quad (12)$$

Therefore the sought parameters of the mean circle center R_0 , e_x , e_y can be determined by minimizing the functional

$$J(e_x, e_y, R_0) = \int_{\Phi} (\Delta R_m(\alpha) - R_0 - e_x \cos \varphi - e_y \sin \varphi)^2 d\varphi \quad (13)$$

When we equate partial derivatives $\frac{\partial J}{\partial R_0}$, $\frac{\partial J}{\partial e_x}$, $\frac{\partial J}{\partial e_y}$ to zero, we obtain

$$\begin{bmatrix} \int_{\Phi} \cos^2 \varphi d\varphi & \int_{\Phi} \cos \varphi \sin \varphi d\varphi & \int_{\Phi} \cos \varphi d\varphi \\ \int_{\Phi} \cos \varphi \sin \varphi d\varphi & \int_{\Phi} \sin^2 \varphi d\varphi & \int_{\Phi} \sin \varphi d\varphi \\ \int_{\Phi} \cos \varphi d\varphi & \int_{\Phi} \sin \varphi d\varphi & \int_{\Phi} 1 d\varphi \end{bmatrix} \cdot \begin{bmatrix} e_x \\ e_y \\ R_0 \end{bmatrix} = \begin{bmatrix} \int_{\Phi} \Delta R_m(\varphi) \cos \varphi d\varphi \\ \int_{\Phi} \Delta R_m(\varphi) \sin \varphi d\varphi \\ \int_{\Phi} \Delta R_m(\varphi) d\varphi \end{bmatrix} \quad (14)$$

3.2. The Case of Any Value of the Coefficient ε

For the given point O' with coordinates (e, δ) , let $R'(\alpha) : \alpha \in A$ be the function describing the profile in the polar system with the center O' . We are looking for such values of e , δ and R_a that the integral

$$J(e, \delta, R_a) = \int_A (R'(\alpha) - R_0)^2 d\alpha \quad (15)$$

reaches the minimum with respect to variables e , δ and R_a . The angle α is a function of the angle φ and depends also on ε , δ , so (see (3-5))

$$\begin{aligned} \alpha &= g(\varepsilon, \delta, \varphi) = \\ &= \arctan(f(\varepsilon, \delta, \varphi) \sin \varphi - e \sin \delta, f(\varepsilon, \delta, \varphi) \cos \varphi - e \cos \delta) \end{aligned} \quad (16)$$

The set A on which profile $R'(\alpha)$ is defined is equal to $A = g(\varepsilon, \delta, \Phi)$ for a certain set Φ . The presented method is iterative in character. Let \hat{e} , $\hat{\delta}$ and \hat{R}_a be certain estimates of unknown values e , δ and R_a . First step is to assume that $\hat{e} = 0$, $\hat{\delta} = 0$ and $\hat{R}_a = R_0$. Let us integrate the equation (15). Since the measured profile is given as a function of the angle φ , the variable in the integral equation (15) will be changed for φ . The dependence between differentials $d\alpha$ and $d\varphi$ will be determined first. If we assume that the roundness deviation is small with respect to the object's radius, from the equations (4) and (5) we obtain the following

$$\sin \alpha = f(\varepsilon, \delta, \varphi) \sin \varphi - \varepsilon \sin \delta \quad (17)$$

$$\cos \alpha = f(\varepsilon, \delta, \varphi) \cos \varphi - \varepsilon \cos \delta \quad (18)$$

Differentiating the equations mentioned above, we obtain

$$\cos \alpha d\alpha = (f_{\varphi}(\varepsilon, \delta, \varphi) \sin \varphi + f(\varepsilon, \delta, \varphi) \cos \varphi) d\varphi \quad (19)$$

$$- \sin \alpha d\alpha = (f_{\varphi}(\varepsilon, \delta, \varphi) \cos \varphi - f(\varepsilon, \delta, \varphi) \sin \varphi) d\varphi \quad (20)$$

Squaring both the equations, then adding and finally extracting square root we obtain

$$d\alpha = \sqrt{f_{\varphi}^2(\varepsilon, \delta, \varphi) + f^2(\varepsilon, \delta, \varphi)} d\varphi = h(\varphi) d\varphi, \quad (21)$$

Using definition given in equation (7), however, one obtains

$$h(\varphi - \delta) = 1 + \frac{\varepsilon \cos(\varphi - \delta)}{\sqrt{1 - \varepsilon^2 \sin^2(\varphi - \delta)}} \quad (22)$$

Changing the variable α for φ in equation (15), we get

$$\begin{aligned} J(e, \delta, R_a) &= \\ &= \int_A (R'(\alpha) - R_0)^2 d\alpha = \int_{\Phi} (R(\varphi) - R_0)^2 h(\varphi - \delta) d\varphi \quad (23) \\ &= \int_{\Phi} (\sqrt{(R_m(\varphi) \cos \varphi - e_x)^2 + (R_m(\varphi) \sin \varphi - e_y)^2} \\ &\quad - R_0)^2 h(\varphi - \delta) d\varphi \end{aligned}$$

where $R_m(\varphi) = R_a + \Delta R(\varphi)$.

Let $e_x = \hat{e}_x + \Delta e_x$, $e_y = \hat{e}_y + \Delta e_y$

$$\text{and } R_a = \hat{R}_a + \Delta R_a \quad (24) - (26)$$

where \hat{e}_x , \hat{e}_y i \hat{R}_a are current evaluations of values e_x , e_y and R_a , whereas Δe_x , Δe_y and ΔR_a are the corrections of these estimations. Let us denote

$$\rho(e_x, e_y, R_a) = \sqrt{(R_m(\varphi) \cos \varphi - e_x)^2 + (R_m(\varphi) \sin \varphi - e_y)^2}$$

Applying the Gauss-Newton minimization algorithm, we obtain,

$$\begin{aligned} \rho(e_x, e_y, \varphi) &\equiv \rho(e_x, e_y, \varphi) + \\ &+ \frac{\partial \rho(e_x, e_y, \varphi)}{\partial e_x} \Delta e_x + \frac{\partial \rho(e_x, e_y, \varphi)}{\partial e_y} \Delta e_y + \\ &+ \frac{\partial \rho(e_x, e_y, \varphi)}{\partial R_a} \Delta R_a \end{aligned} \quad \left. \vphantom{\rho(e_x, e_y, \varphi)} \right|_{(e_x, e_y, R_a) = (\hat{e}_x, \hat{e}_y, \hat{R}_a)} \quad (27)$$

Thus

$$\begin{aligned} J_1(\Delta e_x, \Delta e_y, \Delta R_a) \\ = J(\hat{e}_x + \Delta e_x, \hat{e}_y + \Delta e_y, \hat{R}_a + \Delta R_a) = \\ = \int_{\Phi} (w(\varphi) - v^T(\varphi) [\Delta e_x \quad \Delta e_y \quad \Delta R_a])^2 \hat{h}(\varphi) d\varphi \end{aligned} \quad (28)$$

where,

$$w(\varphi) = \hat{R}(\varphi) - R_o \quad (29)$$

$$v(\varphi) = \frac{1}{\hat{R}(\varphi)} \begin{bmatrix} \hat{R}_m(\varphi) \cos \varphi - \hat{e}_x \\ \hat{R}_m(\varphi) \sin \varphi - \hat{e}_y \\ \hat{R}_m(\varphi) - \hat{e} \cos(\varphi - \hat{\delta}) \end{bmatrix} \quad (30)$$

$$\hat{h}(\varphi) = 1 + \frac{\hat{e} \cos(\varphi - \hat{\delta})}{\sqrt{1 - \hat{e}^2 \sin^2(\varphi - \hat{\delta})}} \quad (31)$$

$$\hat{R}_m(\varphi) = \hat{R}_a + \Delta R_m(\varphi) \quad (32)$$

$$\hat{R}(\varphi) = \sqrt{(\hat{R}_m(\varphi) \cos \varphi - \hat{e}_x)^2 + (\hat{R}_m(\varphi) \sin \varphi - \hat{e}_y)^2} \quad (33)$$

Minimizing the functional J_1 over Δe , $\Delta \delta$ and ΔR_a , we finally get

$$\begin{bmatrix} \Delta e_x \\ \Delta e_y \\ \Delta R_a \end{bmatrix} = \left(\int_{\Phi} v(\varphi) v^T(\varphi) \hat{h}(\varphi) d\varphi \right)^{-1} \int_{\Phi} w(\varphi) v(\varphi) d\varphi. \quad (34)$$

Considering the derived dependences we can define the algorithm enabling us to determine the parameters of the associated circle, e , δ and R_a . The algorithm is iterative and can be described in the following way:

1. Let's assume $\hat{e} = 0$, $\hat{\delta} = 0$ and $\hat{R}_o = R_a$.
2. From the equations (32), (33), (30), (29) and (31) we subsequently determine $\hat{R}_m(\varphi)$, $\hat{R}(\varphi)$ and $v(\varphi)$, $w(\varphi)$, $\hat{h}(\varphi)$.
3. From the relationship (34) we determine corrections Δe_x , Δe_y and ΔR_a of the current estimates \hat{e}_x , \hat{e}_y and \hat{R}_a .
4. We assume that $\hat{e}_x := \hat{e}_x + \Delta e_x$, $\hat{e}_y := \hat{e}_y + \Delta e_y$, $\hat{R}_a := \hat{R}_a + \Delta R_a$.

5. If the stop condition $|\Delta e_x| + |\Delta e_y| + |\Delta R_a| \leq \xi$ is fulfilled for a suitably selected small number ξ , we stop the algorithm. Otherwise we go to step 2 of the algorithm.

It is worth noting that the first step of the defined algorithm corresponds to the traditional method described in step 2.

4. THE ALGORITHM FOR INTERRUPTED PROFILES – THE SIMULATION RESULTS

Let's consider the real profile

$$R'(\alpha) = 1 + k(\sin(3\alpha) + 0.5 \cos(10\alpha) + 0.2 \sin(50\alpha + \pi/2)),$$

$$k = 0.001$$

defined in the angle range $\alpha \in [\alpha_1, \alpha_2] = A$. Let's assume that $\varepsilon = 0.1$, $\delta = \pi/4$, $\alpha_1 = -\pi/2$, $\alpha_2 = \pi/2$.

In the fig. 2, the consecutive samples of the measured profile can be seen.

Using the algorithm described above, we obtained subsequent values of corrections $(\Delta e_x, \Delta e_y, \Delta R_a)$

1. The first iteration:

$$\begin{aligned} \Delta e_x &= 0.0706799, \Delta e_y = 0.0727162, \\ \Delta R_a &= 0.0449252 \end{aligned}$$

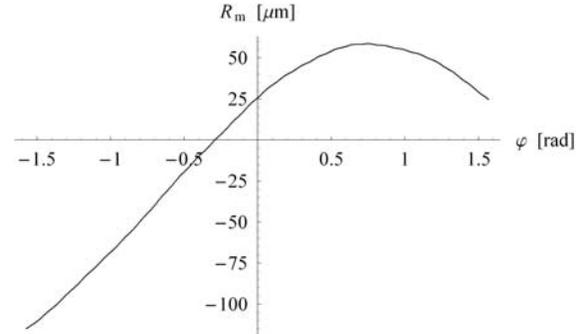


Fig. 2. Samples of the measured profile $R_m(\varphi)$ obtained through a simulation

2. The second iteration:

$$\begin{aligned} \Delta e_x &= 0.0001163442, \Delta e_y = -0.00208172, \\ \Delta R_a &= -0.00239817 \end{aligned}$$

3. The third iteration:

$$\begin{aligned} \Delta e_x &= -3.006325 \cdot 10^{-6}, \Delta e_y = 2.49451 \cdot 10^{-7}, \\ \Delta R_a &= -2.80133 \cdot 10^{-6} \end{aligned}$$

4. The fourth iteration:

$$\begin{aligned} \Delta e_x &= 1.4454 \cdot 10^{-9}, \Delta e_y = -1.45832 \cdot 10^{-9}, \\ \Delta R_a &= 8.57539 \cdot 10^{-10} \end{aligned}$$

In practice, after two iterations we obtain accurate parameters $(\Delta e_x, \Delta e_y, \Delta R_a)$. Fig. 3 shows the deviation of profile $\Delta R'(\alpha)$ after the first and the second iterations have been performed.

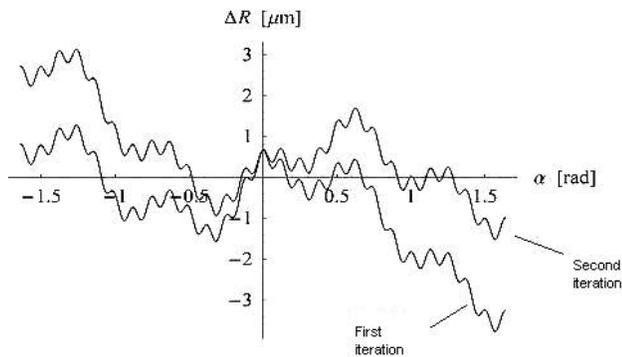


Fig. 3. Profile deviation $\Delta R'(\alpha)$ after the first and the second iteration.

5. CONCLUSIONS

High accuracy roundness deviation measurements are essential in metrology of geometrical quantities. That is why the accurate determination of the associated circle center is so important. The problem has been recognized well for closed roundness profiles. Many machine parts, however, e.g. longitudinal sections profiles of bearing rings, have interrupted (non-closed) profiles. The paper presents the algorithm that makes it possible to calculate the associated circle center by means of the least squares method for interrupted profiles. The condition that the eccentricity-nominal radius ratio is small does not have to be fulfilled.

The proposed concept of calculating the position of the associated circle center was verified with a computer simulation. It confirmed both the correctness of the derived relationships and the possibility to obtain accurate results as early as the second iteration.

6. REFERENCES

- Janecki, D.; Adamczak, S. & Cabaj, M. (2006). Determining the characteristics of sensors displacement in instruments for roundness measurements. XVIII IMEKO World Congress, *Metrology for a Sustainable Development*, Rio de Janeiro, Brazil, September 2006, pp. 17 – 22.
- Rudziński, R.; Dąbrowski, W. & Żebrowska-Łucyk, S. (1995). *System and software improvements for precision metrology equipment used to measure roundness deviation*, PAK, Warszawa, Vol. 2/1995, pp. 30–33.
- Spiegel, M. R. (1971). *Theory and Problems of Advanced Mathematics for Engineers and Scientists*, Schaum's outline series, McGraw-Hill Book Co., New York.
- Whitehouse, D J. (1994). *Handbook of Surface Metrology*. Institute of Physics Publishing, Bristol-Philadelphia.
- Żebrowska-Łucyk, S. (2000). *Non-reference method in rotary elements surface macrogeometry investigations*. Warsaw Technical University. Research Works. Mechanics no. 187, Oficyna Wydawnicza Politechniki Warszawskiej, Warszawa.