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CALCULATION OF THE ASSOCIATED CYLINDER AXIS FOR ELEMENTS MEASURED BY THE "BIRD-CAGE" STRATEGY

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Abstract: *Cylindrical surfaces are common in industrial machining so accurate evaluation of cylindricity is a very important application in geometrical quantities metrology. New GPS standards describe four measuring strategies: a cross-section method, a generatrix method, a point method and so-called "bird-cage" method. The first three methods are relatively well developed. The last one, however, the most accurate method has not been well described mathematically. The paper presents the mathematical model of calculating the cylinder axis orientation with the "bird-cage" strategy.*
Key words: *associated cylinder, axis orientation, "bird-cage" strategy*

1. INTRODUCTION

The usual guideline is that the accuracy of cylindrical surfaces is assessed basing on the roundness profile deviations in a few selected cross sections of the specimen. Since the whole surface affects the mating elements, modern technological processes require a constant monitoring of cylindrical profiles [Osanna et al. (1992)]. The profiles should be evaluated with parameters relating to the whole surface of the element. Parameters and guidelines concerning cylindricity measurement are defined by ISO/TS 12180 standard, a chain link of Geometrical Product Specifications standards. ISO/TS 12180 standard specifies four cylindricity measurement strategies: with a cross-section method, a generatrix method, a "bird-cage" method that is a combination of the two previous methods, and a point method. The "bird-cage" method, the most accurate according to the standard, has not been properly mathematically described yet.

As specified in ISO/TS 12180 standard, cylindricity deviations should be calculated to the so called associated cylinder, that is, an ideal cylinder associated with the measured profile by the specific convention (e.g. with the least squares method or minimum zone method).

The studies carried out so far (e.g. in [Lao et al (2003), Murthy (1992)]) show that the accurate determination of the associated cylinder axis orientation significantly improves the accuracy of the cylindricity measurement.

2. MATHEMATICAL MODEL OF CYLINDRICITY PROFILE

The cylindricity profile can be best described in cylindricity coordinates (φ, z, C) where φ denotes the cylinder rotation angle, z coordinate defines the height, and z is the profile value [Żebrowska-Łucyk (1983)]. So, a random cylindricity profile can be denoted as follows:

$$C(\varphi, z)\,, \tag{1}$$

and

$$0 \le \varphi \le 2\pi \text{ and } 0 \le z \le H\,, \qquad (2)\text{ - }(3)$$

where H is the cylinder height.

Any cylinder axis can be defined with an equation describing coordinates of intersection between the axis and horizontal plane with coordinate z. There are a number of methods to do that.

This paper focuses on the following one:

$$e_x(z) = \alpha_x + \left(\frac{2z}{H} - 1\right)\beta_x\,, \tag{4}$$

$$e_y(z) = \alpha_y + \left(\frac{2z}{H} - 1\right)\beta_y\,. \tag{5}$$

As shown, the intersection points between the axis and planes $z = 0$ and $z = H$ (see Fig. 1) will be equal respectively:

$$(\alpha_x - \beta_x, \alpha_y - \beta_y) \text{ and } (\alpha_x + \beta_x, \alpha_y + \beta_y)\,. \tag{6}$$

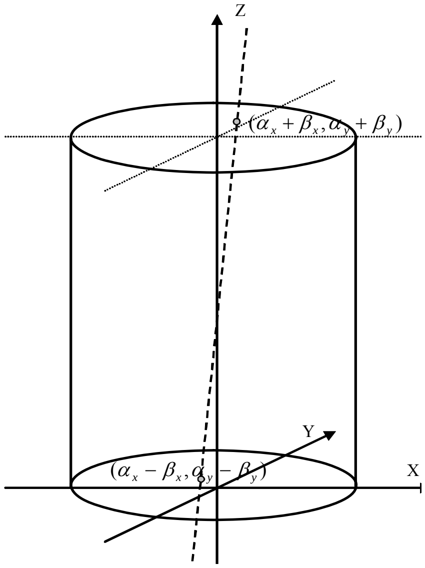

Fig. 1. Orientation and parameters of the true specimen axis in relation to the nominal cylinder

If the specimen axis departures from the sensor rotation axis (table) are small, we can state that

$$C_s(\varphi,z)=C_o+e_x(z)\cos\varphi+e_y(z)\sin\varphi=\theta^T\psi(\varphi,z)\,, \quad (7)$$

where

$$\theta=[\alpha_x \quad \beta_x \quad \alpha_y \quad \beta_y \quad C_0]^T \quad (8)$$

$$\psi(\varphi,z)=\begin{bmatrix}\cos\varphi\\ \left(\frac{2z}{H}-1\right)\cos\varphi\\ \sin\varphi\\ \left(\frac{2z}{H}-1\right)\sin\varphi\\ 1\end{bmatrix} \quad (9)$$

It should be noted that in the adopted method of the cylinder axis parametrization all elements of vector $\psi(\varphi,z)$ are orthogonal in the region $\varphi\in[0,2\pi]$, $z\in[0,H]$. It will let us simplify the derivations.

3. CALCULATING THE ASSOCIATED CYLINDER AXIS ORIENTATION

The „bird-cage" measurement method combines two methods: cross-sections method and generatrix method. Therefore, first, the relationships allowing the identification of the associated cylinder axis orientation will be determined for the cross-sections method and generatrix methods.

3.1. Calculating the associated cylinder axis orientation with the cross-sections method

Figure 1 represents the specimen scanning trajectory obtained by the measuring sensor in the cross-sections method.

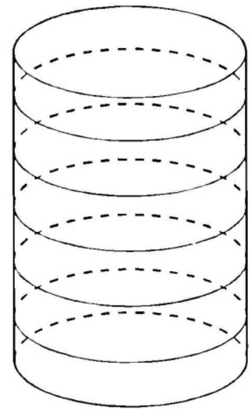

Fig. 2. Cylindricity measurement by the cross-sections method [2]

The cross-sections measurement results in a sequence of samples

$$C(\varphi_i^c,z_j^c)\,,\ i=0,1,...,N_{c-1},\quad j=0,1,...,M_c-1\,, \quad (10)$$

where N_c is the number of test points per cross sections, and M_c is the number of cross-sections [Adamczak et al. (2002)].

We assume that in the cross-section the samples are collected uniformly with the fixed sampling interval, i.e.

$$\varphi_i^c=\frac{2\pi i}{N_c}\,,\ i=0,1,...,N_c-1\,. \quad (11)$$

However, the distances between the cross-sections do not have to be equal.

Based on dependences (4)-(10) we can define the index:

$$J_c(\theta)=\frac{1}{2}\sum_{j=0}^{M_c}\sum_{i=0}^{N_c-1}(C(\varphi_i^c,z_j^c)-\theta^T\psi(\varphi_i^c,z_j^c)^2\,. \quad (12)$$

When partial derivatives $\frac{\partial J}{\partial\theta_i}$ are determined and equated to zero, we obtain:

$$A_c\theta=b_c\,, \quad (13)$$

where

$$A_c=\begin{vmatrix}\frac{N_c}{2}D^c & 0 & 0\\ 0 & \frac{N_c}{2}D^c & 0\\ 0 & 0 & N_cM_c\end{vmatrix}, \quad (14)$$

$$D^c=\begin{vmatrix}M_c & \sum_{j=0}^{M_c-1}\left(\frac{2z_j^c}{H}-1\right)\\ \sum_{j=0}^{M_c-1}\left(\frac{2z_j^c}{H}-1\right) & \sum_{j=0}^{M_c-1}\left(\frac{2z_j^c}{H}-1\right)^2\end{vmatrix}, \quad (15)$$

$$b_c = \begin{vmatrix} \frac{N_c}{2}\sum_{j=0}^{M_c-1} e_x(z_j^c) \\ \frac{N_c}{2}\sum_{j=0}^{M_c-1}\left(\frac{2z_j}{H}-1\right)e_x(z_j^c) \\ \frac{N_c}{2}\sum_{j=0}^{M_c-1} e_y(z_j^c) \\ \frac{N_c}{2}\sum_{j=0}^{M_c-1}\left(\frac{2z_j}{H}-1\right)e_y(z_j^c) \\ \sum_{i=0}^{N_c-1}\sum_{j=0}^{M_c-1} C(\varphi_i^c, z_j^c) \end{vmatrix} \tag{16}$$

where $(e_x(z_j^c), e_y(z_j^c))$ stand for evaluations of the cylinder true axis, i.e.

$$e_x = \frac{2}{N_c}\sum_{i=0}^{N_c-1}\cos\varphi_i^c C(\varphi_i^c, z_j^c)\,, \tag{17}$$

$$e_y = \frac{2}{N_c}\sum_{i=0}^{N_c-1}\sin\varphi_i^c C(\varphi_i^c, z_j^c)\,. \tag{18}$$

Dependences (13) - (18) allow calculation the associated cylinder axis orientation in the cross-sections method measurement [Adamczak et al (2003)].

3.2. Calculating the associated cylinder axis orientation with the generatrix method

Figure 3 shows the specimen scanning trajectory obtained by the measuring sensor in the generatrix method.

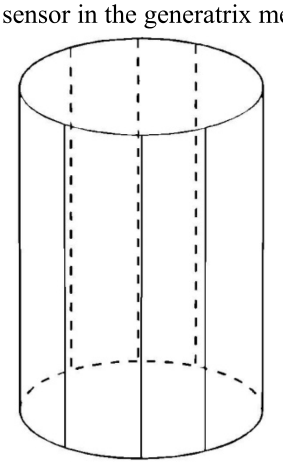

Fig. 3. Cylindricity measurement by the generatrix method [2]

The generatrix method measurement results in a sequence of samples

$$C(\varphi_i^l, z_j^l) \quad i = 0,1,\ldots,N_l-1, \quad j = 0,1,\ldots,M_l-1\,, \tag{19}$$

where: N_l stands for the number of longitudinal sections, and M_l stands for the number of samples per longitudinal section. We assume that the samples in the longitudinal sections are collected uniformly in the range $z^l \in [0, H]$, so

$$z_j^l = \frac{Hj}{M_l-1}, \quad j = 0,1,\ldots,M_l-1 \tag{20}$$

In order to calculate the coordinates of the associated cylinder axis orientation, the index was defined:

$$J_l(\theta) = \frac{1}{2}\sum_{j=0}^{M_l}\sum_{i=0}^{N_l-1}(C(\varphi_i^l, z_j^l) - \theta^T\psi(\varphi_i^l, z_j^l))^2 \tag{21}$$

Basing on the necessary conditions of optimality $\frac{\partial J_l}{\partial \theta_i} = 0$ we obtain

$$A_l\theta = b_l\,, \tag{22}$$

where

$$A_l = \begin{vmatrix} M_l s_3 & 0 & M_l s_5 & 0 & s_1 \\ 0 & \gamma_l s_3 & 0 & \gamma_l s_5 & 0 \\ M_l s_5 & 0 & M_l s_4 & 0 & s_2 \\ 0 & \gamma_l s_5 & 0 & \gamma_l s_4 & 0 \\ s_1 & 0 & s_2 & 0 & M_l N_l \end{vmatrix}, \tag{23}$$

$$b_l = \begin{bmatrix} M_l s_1 a(\varphi_i^l) \\ \gamma_l s_1 b(\varphi_i^l) \\ M_l s_2 a(\varphi_i^l) \\ \gamma_l s_2 b(\varphi_i^l) \\ \sum_{i=0}^{N_l-1}\sum_{j=0}^{M_l-1} C(\varphi_i^l, z_i^l) \end{bmatrix}. \tag{24}$$

Relationships (23)-(24) were formulated with:

$$\sum_{j=0}^{M_l-1}\left(\frac{2z_j^l}{H}-1\right) = 0 \tag{25}$$

and the following denotation

$$\sum_{j=0}^{M_l-1}\left(\frac{2z_j^l}{H}-1\right)^2 = \frac{{M_l}^2 + M_l}{3M_l - 2} = \gamma_l\,. \tag{26}$$

The corresponding values of $s_{1\text{-}6}$ from relationships (23) and
(24) can be expressed as:

$$s_1 = \sum_{i=0}^{N_l-1}\cos\varphi_i^l,\ s_2 = \sum_{i=0}^{N_l-1}\sin\varphi_i^l,\ s_3 = \sum_{i=0}^{N_l-1}\cos^2\varphi_i^l, \\ s_4^l = \sum_{i=0}^{N_l-1}\sin^2\varphi_i^l,\ s_5^l = \sum_{i=0}^{N_l-1}\sin\varphi_i^l\cos\varphi_i^l. \tag{27}$$

Values $a(\varphi_i^l)$ and $b(\varphi_i^l)$ from dependences (24) are the parameters of a regression line $a + b\left(\frac{2z}{H}-1\right)$ for the *i*-th longitudinal section, thus

$$a(\varphi_i^l) = \frac{1}{M_l}\sum_{j=0}^{M_l-1} C(\varphi_i^l, z_j^l)\,, \tag{28}$$

$$b(\varphi_i^l) = \frac{1}{\gamma_l}\sum_{j=0}^{M_l-1} C(\varphi_i^l, z_j^l)\left(\frac{2z_j^l}{H}-1\right). \tag{29}$$

3.3. Calculating the associated cylinder axis orientation with the "bird-cage" method

In the "bird-cage" measurement (see Fig.4) the obtained cylindricity profile is a combination of profiles obtained by the cross-sections and generatrix methods.

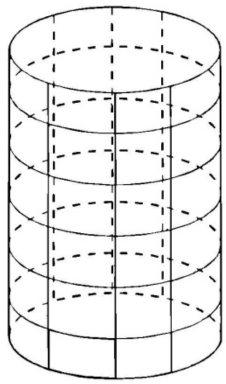

Fig. 4. Cylindricity measurement by the "bird-cage" method [2]

The index used to calculate the associated cylinder parameters for the „bird-cage" method is a sum of the index for the cross-sections method (described by (12)) and the index for the generatrix method (described by (21)). Thus, it can be expressed as:

$$J(\theta)=\frac{1}{2}\left(\sum_{j=0}^{M_c}\sum_{i=0}^{N_c-1}(C(\varphi_i^c,z_j^c)-C_0-\theta^T\psi(\varphi_i^c,z_j^c)^2 + \sum_{j=0}^{M_l}\sum_{i=0}^{N_l-1}(C(\varphi_i^l,z_j^l)-C_0-\theta^T\psi(\varphi_i^l,z_j^l)^2\right). \tag{30}$$

When partial derivatives $\frac{\partial J}{\partial \theta_i}$ are determined and equated to zero, we obtain:

$$(A_c + A_l)\theta = b_c + b_l \,, \tag{31}$$

where A_c, b_c, A_l, b_l were described by dependences: (14), (16), (23) and (24) respectively.

Relationships (8), (14), (16), (23), (24) and (31) constitute the complete mathematical model for calculating the associated cylinder parameters with the "bird-cage" method.

4. EXAMPLE OF APPLICATION

Developed relationships were be used in the experimental part of work on calculating cylindricity parameters in „bird-cage" measurements.

Figure 5 shows the diagram of the cylindrical surface extracted by the "bird-cage" method without calculation of the associated cylinder axis.

Accordingly, Fig. 6 shows the same cylindrical surface after calculation the associated cylinder axis. As one can see, application of described algorithm there is a significant difference between compared surfaces.

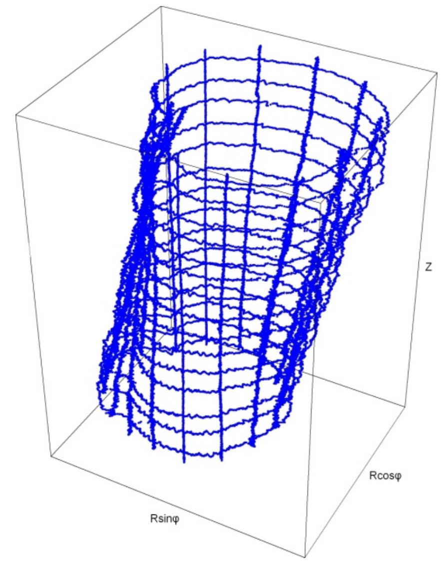

Fig. 5. Cylindrical surface extracted by the "bird-cage" method without calculation of the associated cylinder axis

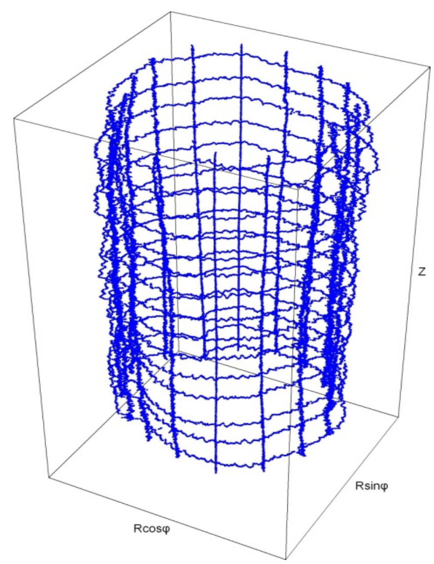

Fig. 6. Cylindrical surface extracted by the "bird-cage" method after calculation of the associated cylinder axis

5. CONCLUSIONS

The PN-EN ISO 12180 standard recommends the „bird – cage" technique as the most accurate strategy for cylindricity measurements, providing accurate data on the tested cylindricity profile. This technique, however, is quite complex. Apart from being time and work-consuming, it requires a very sophisticated mathematical description. The analysis of relationships included in section 3 makes the problem more visible. Matrices which make it possible to identify the orientation of the associated cylinder axis by the „bird-cage" method are a sum of respective matrices for the cross-sections and generatrix methods.

Developed relationships were used in practice for calculation cylindricity parameters in „bird-cage" measurements.

Positive completion of the experimental part of work proved that developed relationships are correct.

It should be noted that for the special case, where in the "bird-cage" measurement both cross-sections and longitudinal sections coordinates are uniformly distributed, the matrices A_c and A_l become diagonal and have the following form:

$$A_c = diag\begin{bmatrix} \frac{N_c M_c}{2} & \frac{N_c \gamma_c}{2} & \frac{N_c M_c}{2} & \frac{N_c \gamma_c}{2} & N_c M_c \end{bmatrix}, \quad (32)$$

$$A_l = diag\begin{bmatrix} \frac{N_l M_l}{2} & \frac{N_l \gamma_l}{2} & \frac{N_l M_l}{2} & \frac{N_l \gamma_l}{2} & N_l M_l \end{bmatrix}. \quad (33)$$

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