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Phyics of Wave Group Velocities in Metrology for Dispersion Media

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Abstract: Phase and group velocities of wave are most frequently used in the research of physical processes and phenomena in dispersion media. The difference existing between phase velocities of ideal harmonic waves and real non-monochromatic waves comes into conflict with their physical interpretation, that sometimes brings some additional errors both in theoretical investigations and the results of physical measurements. The concepts developed at the beginning of the 20th century have not been properly estimated and now they need to be specified and revised.

Keywords: phase and group velocity.

Determination of different wave's propagation velocities in dispersion media has both scientific and practical significance, as it is used in investigation of the wide array of physical phenomena and high-precision physical-and-technical measurements [1-2]. As we know, dispersion media are those whose refractive index depends on the wave frequency. For the light waves these are the following: air, liquids, glass; for radio waves – ionized plasma, etc. There exist dispersion media for acoustic, mechanical, electric and many other types of oscillations [2].

In modern physics literature phase- and group- wave's velocities in dispersion media and consequently their refractive indices are equally used for both qualitative and quantitative estimation of physical phenomena and the parameters to be measured [1-7]. The present paper shows whether the use of phase velocity for estimation of experimental data and interpretation of real physical processes and phenomena in dispersion media is correct. It is known that phase velocity exists for only strictly monochromatic waves over the infinitely large time range [1-2], whereas real waves don't comply with the condition of monochromatic. In dispersion media real waves are modulated in time by random processes, function switching–on and–off function and the discrete character of radiation, for example, as light quanta. Hence it follows that there are no ideal harmonic (monochromatic) waves in nature and thus there is no phase velocity in dispersion media. The use of phase velocity for real waves is justified for non-dispersion media only.

In physics textbooks, including special literature on wave's theory [2-3], to describe physical processes of wave's reflection and refraction in dispersion media the concept of phase refractive index is used. Sometimes the term "refractive index in dispersion media" is used without specifying if it adheres to phase or group velocity [5-6]. The author considers that it results from the contradicting statements of these terms in home and foreign literature that disorients the reader. For example, for a long time it has been stated in reference books on physics and technology that any real waves are not monochromatic and propagate in dispersion media with only group velocity [1]. But the same books when dealing with theory and solution of practical problems refer to phase velocities as actually existing ones.

The value of the phase refractive index for the phase velocity of electromagnetic waves may be derived from the known wave equation for electromagnetic waves [1-4]

$$\nabla^2 E = \frac{\mu\varepsilon}{c^2} \cdot \frac{\partial^2 E}{\partial t^2} \tag{1}$$

where ∇ - is Laplace operator; t is time; E is electromagnetic wave electric field strength ; μ and ε are medium magnetic permeability and permittivity; c is light velocity in vacuum.

For media homogeneous in space and time, the equation (1) has a solution [1-2] in the form of an ideal harmonic wave

$$E(t) = E_1 \cos(\omega t + \varphi_0) = E_1 \cos\psi(t)$$

where E_1 is an amplitude of electromagnetic waves; ω is angular frequency; φ_0 is initial phase, $\psi\ (t)$ - total phase.

In this case phase velocity υ_p of a harmonic electromagnetic wave is derived from the equation

$$\upsilon_p = \frac{c}{\sqrt{\varepsilon\mu}} = \frac{c}{n_P} \tag{2}$$

where n_P is a wave-phase refractive index.

Actually real waves are not harmonic signals and dispersion media are not homogeneous in space and time; the waves in these media propagate with only group velocity [1] and their propagation and refraction directions are set by a group refractive index. Unfortunately these statements have not been properly estimated and used in the theory and practice of wave's propagation both in classical physics and adjacent scientific and technical disciplines [2-4,7-10].

Propagation of modulated waves in dispersion media is usually described with phase velocity being used for carrier waves, and group velocity for modulated waves envelopes [2-10]. This statement substantially misrepresents both the real physical picture of the process and phenomena under study and interpretation of physical and technical measurements results. Thus in Ref. [4] it is stated that the phase velocity of Gaussian beam in optical waveguides is greater than that of the light in vacuum. In fact both carrier waves and modulating waves in dispersion media propagate with only group velocities not exceeding the speed of light in vacuum.

Let's dwell upon the arguments in support of the real signal presentation as random modulated oscillation. As a vivid example of it we may take record stabilized laser radiation which is used as quantum frequency standard [8]. In this case laser radiation may be considered as the closest one to ideal monochromatic radiation. Its relative frequency instability may be estimated as the magnitude of an order $\frac{\Delta\omega}{\omega_0} \cong 10^{-15}$, where $\Delta\omega$ is frequency drift; ω_0 is mean value of laser frequency per a time period set for long-time or short-time frequency stability. The present frequency drift (quantum frequency standard) means that laser frequency $\omega(t)$ is a random continuously changing function, and thus, laser radiation is always modulated in frequency by a random process. Since instantaneous radiation frequency $\omega(t)$ is determined as a derived function of signal $\psi(t)$ in time $\omega(t) = d\psi(t)/dt$, then frequency variation will result in phase modulation by a random law. Random phase changes in their turn will affect radiation amplitude changes resulting in random amplitude modulation. Thus, a real optical wave evens the stabilized one is always modulated at the same time in amplitude, frequency and phase by random processes. That is why it propagates in dispersion medium with only group velocity [1, 11].

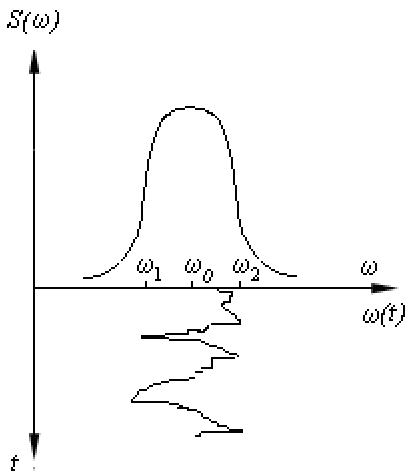

Fig.1: Spectrum density of laser field $S(\omega)$ with average frequency ω_0 under random variation of the resulting frequency $\omega(t)$ over time *t*.

The electric field of laser radiation *E(t)* may be presented as a narrowband (or quasimonochromatic) random process or in a simplified way as quasiharmonic oscillation

$$E(t) = E_0(t)\cos[\omega(t)\cdot t + \varphi_0(t)] = E_0(t)\cos\psi(t) \tag{3}$$

where $E_0(t)$ is electric field amplitudes and $\omega(t)$ is angular frequency; $\varphi_0(t)$ is initial phase; $\psi(t)$ is total phase, respectfully. These are random changes functions.

Equ. (3) is not the only form of representing real quasimonochromatic waves. In a number of cases a more convenient form may be used a harmonic one. Such approach is possible when optical wave (3) is a stationary random process and averaging is performed over the photo detector time constant which is many times as great as the light wave period. Then in some cases a more usual formula may be used for narrowband light waves

$$E(t) = E_0\cos(\omega_0\cdot t + \varphi_0) \tag{4}$$

where $E_0 = \overline{E_0(t)}$, $\omega_0 = \overline{\omega(t)}$ and $\varphi_0 = \overline{\varphi_0(t)}$ are values averaged over time of amplitude, angular frequency and phase, respectfully. It means that wave (4) also propagates in dispersion media with group velocity and its characteristics are received from statistically averaged parameters of a random wave.

For the spectral analysis of a real electric field *E(t)* in the form of eq. (3) we can use complex amplitude spectrum density $S(\omega)$ obtained over a reasonably large time interval, using inverse Fourier transform [4]

$$S(\omega) = \int_{-\infty}^{+\infty} E(t)\, \mathrm{e}^{-\mathrm{j}\omega t}\, dt \tag{5}$$

where $\mathrm{j} = \sqrt{-1}$. Amplitude spectrum $S(\omega)$ characterizes continuous amplitude distribution of a random process $E(t)$ depending on frequency ω. It means that frequency variations $\omega(t)$ result in random variations of radiation spectrum amplitudes $S(\omega)$ which are small for the certain frequency value ω, but different in amplitudes for each frequency value ωwithin the limits of $\omega_1 < \omega < \omega_2$ (see fig.1).

The fundamental importance of this paper lies in the statement that each component of a real random wave and a modulated wave with frequency ω propagates with group velocity and not with the phase one, as it is stated in relevant literature. Otherwise we could have obtained ideal monochromatic radiation from a random signal, but it is impossible as the filter with bandwidth equal to zero is physically unrealizable. Spectral components $S(\omega)$ are also modulated in their random laws for each concrete frequency ω in band $\Delta\omega = \omega_2 - \omega_1$.

Spectral components of the pulse over the frequency band $\Delta\omega$ will propagate with group velocities $\upsilon_g(\omega)$ in the same way as a pulse envelope which changes relatively slowly.

The above mentioned statements may be applied to the waves of any nature, propagating in their dispersion media with group velocities. Now we are going to consider, how the author's approach may be applied to the solution of certain problems.

1. Radio waves propagation in ionosphere

Ionosphere is the most vivid one to reveal physical absence of phase velocity as a real parameter. In this case, when only group velocities of real radio waves and their spectral components are used in dispersion ionosphere with abnormal refraction, the following advantages are revealed. There is no need to explain that the phase velocity of radio waves in ionosphere "may exceed the velocity of light in vacuum" [2, 9-10,12]. Concerning this question, a paradoxical situation arises in relevant literature, when first the concept of phase velocity is artificially introduced for monochromatic radio wave in dispersion medium, which actually can't exist. Then it is stated that it can propagate with the velocity exceeding that of the light in vacuum. We know that any wave transfers energy even if it is a monochromatic one, but the velocity of its transfer cannot exceed the velocity of light in vacuum. Therewith it is explained that the phase of monochromatic wave propagates with this velocity, though it is the phase that specifies the value of the wave amplitude. As a matter of fact, as there is no any phase velocity, there can't be any waves propagating with the velocity exceeding that of light in vacuum.

However the author does not deny using phase velocity in dispersion media as a hypothetic parameter for purely theoretical investigations when there is no way to do without it. In this case it is necessary to point out that the introduced parameter is merely theoretical. For example, it is practically impossible to determine phase velocity in dispersion medium, but it can be calculated by the theoretical model [2] or from the value of the group velocity, say, from the known equation $n_p\, n_g$=1. It may be required for testing the validity of theoretical models of dispersion media against their real models. Yet the derivation of the group velocity from the theoretical values of phase velocity is not accurate enough, and in practice, for this purpose the results of experimentally determined values of group velocities are used as well as the dispersion methods of measuring group mean-integral refractive index of ionosphere [7, 9-10,12].

2. Optical waves propagation in troposphere

At present phase and group refractive indices of optical waves in atmosphere are widely used in physics [12], optics [1-4,8], astronomy, geodesy [12-14] and other fields of science and technology to account for the dispersion medium effect on the results of angular and linear measurements, in special guidance systems of laser and optical radiation as well as in other fields.

Special attention should be paid to the following applications of the phase refractive index:

(1) in interferometers for measuring distances and interferometers-refract meters for medium refractive index determination;

(2) in dispersion angular refractometry and for determination of refraction corrections by interference techniques [7,14].

The group refractive index is used to determine the light propagation velocity in modulation- pulse- and phase laser range-finding, and dispersion refractometry using external modulation [7,12,14].

Analysis of the available literature shows that at present the author of all the basic works on determination of the air refractive index for optical waves is a Swedish scientist B. Edlen [5-6]. However in his works he uses only the term “refractive index” and does not specify what kind of refractive index (phase or group) his formulas are derived for. In technical literature the value of the refractive index determined by B. Edlen was renamed into the “phase refractive index” due to the substitution of the real quasimonochromatic radiation by an ideal monochromatic one. This resulted from the erroneous statement of the phase velocity existence in dispersion medium. Since B. Edlen derived the value of the refractive index from interferometer measurements, using real radiation, which (as shown above) propagated with group velocity, he practically determined the group refractive index and not the phase one as it is commonly assumed in the existing literature [5-7,14] .

As a result, the situation arises in which the substitution of the term “phase refractive index” by the “group refractive index” does not make B. Edlen’s refractive index incorrect, when used for determination of the velocity and the direction of optical waves, to solve the above mentioned problems. The given paper puts forward the comparative analysis of the theoretical research [11] and the experimental data of the two-wave optical range findings presented in literature [13,14]. By solving the problem of determining the true velocity of optical waves in the atmosphere we can considerably improve the accuracy of both optical range finders and more precise instruments, refractometric range finders. The latter were created in 1970s of the last century on the basis of the two- and multi-wave dispersion methods and are seldom used now.

To solve the set problem we should consider the formula, referred (R) to (Ref. [11]) as the group index of optical waves tropospheric refractive index under the standard conditions, in the form of Cauchy equation

$$(n_0^{\mathrm{R}} - 1)10^6 = 272.612\,9 + \frac{1.529\,4}{\lambda^2} + \frac{0.01367}{\lambda^4}\ , \qquad (6)$$

where n_0^{R} is a group refractive index of optical waves in standard conditions. Under the standard conditions the following are meant: T_0 =288.15^{0}K is temperature, ^{0}K (t_0=15^{0}C); P_0 =760 mmHg is pressure; humidity is "dry air" with 0.03% of carbon dioxide, λ - is the wavelength in vacuum, µm.

Eq. (6), previously referred to as the phase refractive index, resulted from the experimental measurements. As any real radiation used for measurements is not, strictly speaking, a monochromatic one existing in the optical radiation carrier-band $\Delta\omega$, it propagates in the atmosphere at the group velocity [5-6] only. From here on Eq. (6) will be referred to as formula of the group refractive index.

For standard conditions the now used (U) formula of the group refractive index n_0^{U} of optical waves in troposphere takes the form of

$$(n_0^{\mathrm{U}} - 1)\cdot 10^6 = 272.6129 + \frac{3\cdot 1.5294}{\lambda^2} + \frac{5\cdot\ 0.01367}{\lambda^4} \qquad (7)$$

Formula (7) was derived by substituting equation (6), as the phase refractive index, into the known Rayleigh-equation, connecting the group and phase refractive indices [7,14].

For the conditions other than the standard, the group refractive index *n* of the air may be calculated by the value of $n_0 = (n_0^{\mathrm{U}}$ or $n_0^{\mathrm{R}})$ in the following expression [7,14]

$$(n^{\mathrm{R(U)}} - 1)\cdot 10^6 = (n_0^{\mathrm{R(U)}} - 1)\cdot 10^6 \cdot \frac{T_0\cdot P}{P_0\cdot T} - \left(17.045 - \frac{0.5572}{\lambda^2}\right)\cdot\frac{e}{T},$$

where *P, T* and *e* are the corresponding mean values of pressure, temperature and humidity measured at the ends of the line at the moment of observation.

The above set problem was solved by comparing the results of the calculations from Eqs. (6) and (7) with the experimental measurements data. The lack of phase velocity and inaccuracy of Eq. (7), widely used now, are most vivid, when comparing the path-length difference calculated for the light wavelengths λ_1 and λ_2 with real measurements by the two-wave optical refractometric range-finder, presented in papers [14]. The instrument employed two lasers with wavelengths λ_1=0.6328 µm and λ_2=0.4416 µm of the red and blue light beams, respectively. In simultaneous measurements of length differences $\Delta_m = D(\lambda_2) - D(\lambda_1)$ high frequency laser-beam modulation (of the order of 2.6 GHz) with wavelengths λ_1 and λ_2 exhibited instrumental error σ of the fractions of millimeter.

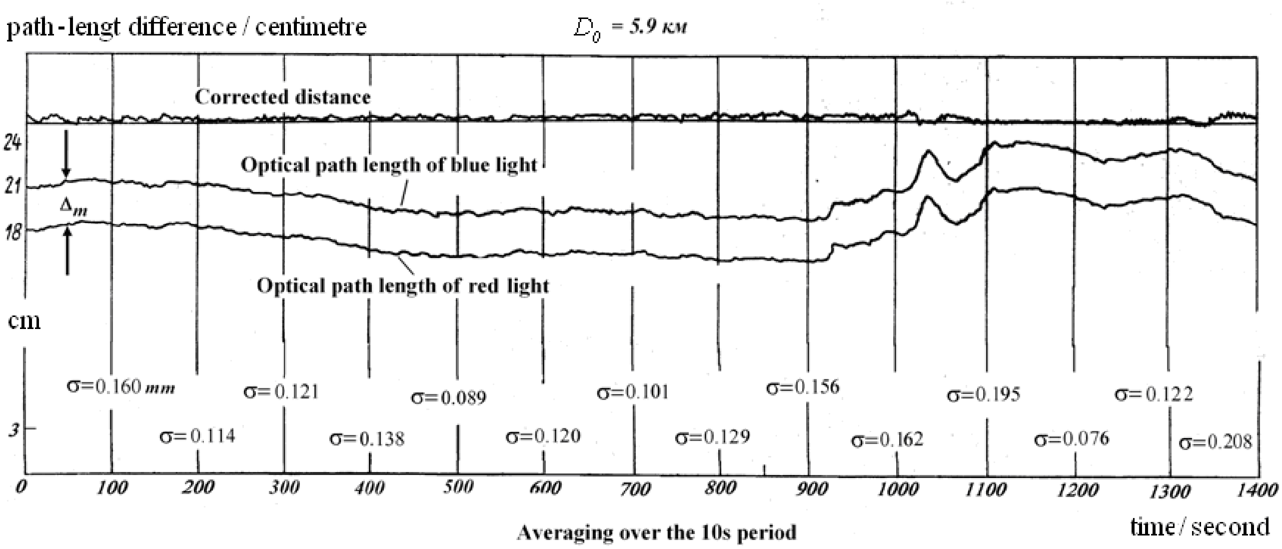

Fig. 2: The data of the field measurements of the line D_0=5.9 km, made by the two-wave phase laser rangefinder K. Earnshaw [13] and J. Berger [14]

The distances $D(\lambda_1)$ and $D(\lambda_2)$ in the table 4 were determined by the formulas: $D(\lambda_1) = D_0 \cdot n(\lambda_1) = D_0[1 + \mathrm{N}(\lambda_1)\cdot 10^{-6}]$, $D(\lambda_2) = D_0 \cdot n(\lambda_2) = D_0[1 + \mathrm{N}(\lambda_2)\cdot 10^{-6}]$ taking into account weather conditions under which measurements with refractometric range finder [13] were taken.

To evaluate the efficiency of the formulas applied (7) and offered (6) by the author, calculations of the refractive indices and the length differences of the lines under measurement have been conducted. They are presented in Tab. 1.

Tab. 1. Results of calculations d by offered (6) and applied (7) formulas

Formula type	λ_1 =0,632 8 μm	
	$N(\lambda_1)$	$D(\lambda_1)$
R	277.060	5 901.635 m
U	285.039	5 901.682 m
	λ_2 =0.441 6 μm	
	$N(\lambda_2)$	$D(\lambda_2)$
R	281.305	5 901.660 m
U	298.428	5 901.760 m
	Path length differences: Δ_d =$D(\lambda_2)$- $D(\lambda_1)$	
R	0.025 m	
U	0.079 m	

The results of the calculations made in the table were compared with the data of the practical measurements made in NOAA by K. Earnshaw [13] and J. Berger [14], presented in Fig. 2.

Following the comparative analysis of the table data and the measurements results presented in Fig. 2, the value of the wave path- length difference (25.0 mm), calculated from the Eqs. offered by the author Eq. (6) is almost the same as the averaged measurement value $\Delta_m \approx 30$ mm. Eq. (7) used by the refractometer creators gives the value of path-length difference equal to 79.0 mm, that is three times as great as that received in practice. Hence it follows that Eq. (7) used nowadays prevent from both obtaining true measurements results and wide introduction of refractometric range-finders into the practice of modern precision geodetic measurements. In a similar way the problem may be considered, as concerning light propagation in glass and other dispersion media with normal refraction wherein optical radiation also propagates with group velocity.

It is very important to use group velocities in dispersion media of quantum frequency standards with record stability of synchronized modes, created due to the periodic sequence of femtosecond laser pulses [8]. In such standards the stabilized frequency spectrum is made by harmonics of femtosecond pulse-periodic sequence envelope. On the basis of such frequency standards the standard of length is planned to be made.

3. Reflection and refraction of light

In the currently available literature these problems are using phase velocities and phase refractive indices [2-4]. In consideration stated these questions taking into account real group velocities of dispersion waves. Let these media be air and glass, where waves propagate with group velocities υ_{g1} and υ_{g2}. Let the boundary of the two plane homogeneous dispersion media 1 and 2 goes along axis x, and the plane wave strikes normal z at angle θ_1 and divides into two: the reflected wave and the refracted one (see fig.3).

Using the known formulae [2-4] we can write the laws of reflection and refraction as follows

$$\frac{\sin\theta_1}{\upsilon_{g1}} = \frac{\sin\theta_2}{\upsilon_{g2}} = \frac{\sin\theta_0}{\upsilon_{g1}} \tag{8}$$

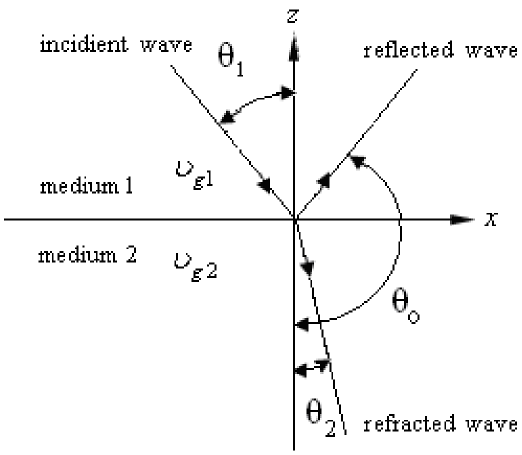

Fig. 3: Reflection and refraction of waves at the two media interface

where $\theta_o = 180^0 - \theta_1$.

Taking into account that radiation in dispersion media propagates with group velocity we derive

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{\upsilon_{g1}}{\upsilon_{g2}} = \frac{n_{g2}}{n_{g1}} \qquad (9)$$

Thus, the directions of real waves when reflected or refracted in dispersion media are determined by the group refractive indices and not the phase ones [2-4].

4. Atomic physics

As the velocity of quantum-mechanical particles presented in de Broglie waves is determined by the velocity of wave packet [6], then as in the previous case, each spectral component will also propagate with the group velocity and not with the phase one. The aggregate of the spectral components propagating with group velocities will give (in the final form) a real velocity of the wave packet. If need be to derive the theoretical value of phase velocity υ_p, it can be obtained from the relation $\upsilon_p \cdot \upsilon_g = c^2$, as for the radio waves in ionosphere.

To a certain degree it concerns the methods of presenting Vavilov and Cherenkov theory of radiation [15]. When the latter is discussed in the relevant literature, the term "the phase velocity of light in dispersion medium" is used.

In conclusion it should be said that the examples mentioned above do not cover all the possible problems the considered approaches could be applied to.

Conclusions

(1) For the description of physical processes of real wave's propagation in the dispersion media, the group velocity and the group refractive index should be used.

(2) Any experimental measurements of waves and their spectral components in dispersion media deal with their group velocities.

(3) In physics and technical disciplines the concepts of "phase velocity" and "phase refractive index" in dispersion media should be considered as having no real basis and should be replaced by "group velocity" and "group refractive index" respectfully. Phase velocity in dispersion media may be used only for theoretical investigations and as an auxiliary parameter of an ideal harmonic wave.

(4) Interpretation of any results of physical-and-technical measurements in dispersion media should deal with the concepts of "group velocity" and "group refractive index".

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