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PERFORMANCE OF ADAPTIVE MONTE CARLO PROCEDURES USING DIFFERENTIAL PRESSURE MEASUREMENT MODEL

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Abstract:
Adaptive Monte Carlo procedure given in Supplement 1 to the "Guide to the expression of uncertainty in measurement" (GUM S1) [1] and a modified GUM S1 procedure [2], [3] were applied to reach a prescribed numerical accuracy of the Monte Carlo results for a chosen confidence level. We assess the performance of the adaptive procedures by employing the measurement model that defines differential pressure generated by a pair of pressure balances. A large number of premature calculation terminations when applying the GUM S1 adaptive procedure leads to a low success rate. Applying the modified GUM S1 shows a reduction in premature termination and improves the success rate.

Keywords: Adaptive Monte Carlo method; differential pressure; pressure balance

1. INTRODUCTION

Due to its stochastic nature, a Monte Carlo Method (MCM) will provide slightly different results of calculation when frequently applied to the same model of measurement. This discrepancy raises the question about the numerical quality of the results. One of the ways to assess the numerical quality is by repeating the entire calculation many times and determining the 'success rate' as the proportion of such calculations that produce results of required accuracy [2]. To control the quality of the results, Supplement 1 to the "Guide to the expression of uncertainty in measurement" (GUM S1) [1] introduces the adaptive Monte Carlo method (AMCM).

In this study, we assess the performance of the procedures of AMCM by employing the differential pressure measurement model. The AMCM procedures are used to achieve the specified accuracy of the Monte Carlo result (the estimate of the differential pressure, and the left- and right-hand endpoints) for a confidence level of $1 - \alpha$ ($\alpha = 0.05$).

2. DIFFERENTIAL PRESSURE SYSTEM

The method for the measurement of differential pressure, used in our study, requires two pressure balances (Figure 1). The method consists of two steps [4], [5], [6], [7]. In the first step, the two pressure balances are initially cross-floated and adjusted at equilibrium. In the second step, pressure balances are separated with the bypass valve, an additional mass Δm is added to the calibrated pressure balance, and pressure is raised until the appropriate float of the pressure balance is achieved.

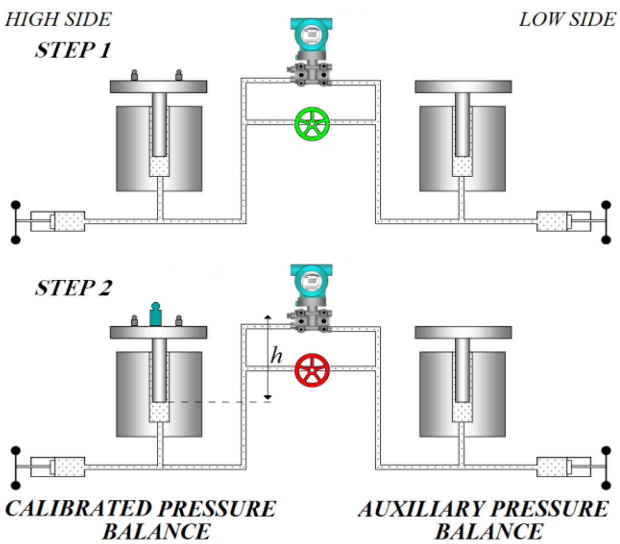

Figure 1: Schematic of the procedure for generating differential pressure. The green valve in step 1. depicts an open bypass valve, and the red valve in step 2. marks a closed bypass valve. The grey color represents the weights used to set the equilibrium state, and the blue weight marks the additional mass Δm.

The system used in this study consists of:
- two oil pressure balances with 4,9 mm^2 effective area of piston-cylinder assemblies made of tungsten carbide,
- float position sensors for measuring piston fall rates,
- trimming weights,

- platinum resistance thermometer built into the pressure balances mounting post for measuring temperatures of the piston-cylinder assemblies.
- combine pressure, humidity, and temperature transmitter for measuring ambient conditions (p_a, RH_a, and t_a),
- six sensors installed in the vicinity of the system for measuring temperature and humidity inhomogeneity in the laboratory,
- connection lines and valves.

3. MODEL OF MEASUREMENT

Detail measurement procedure and calculation of differential pressure generated *via* two pressure balances has been studied and described in [6]. Differential pressure is calculated as

$$\Delta p = \frac{\Delta m\, g\left[1-(\rho_a-1.2)\left(\frac{1}{\rho_m}-\frac{1}{8000}\right)\right]\left(1-\frac{\rho_a}{8000}\right)}{A_{0h}\left(1+\lambda_h(p_i+\Delta p_n)\right)\left(1+2\alpha_h(t_h-t_{\mathrm{ref}})\right)}$$
$$+p_i\left[\frac{(1+\lambda_h p_i)\left(1+2\alpha_h(t_{ih}-t_{ref})\right)}{\left(1+\lambda_h(p_i+\Delta p_n)\right)\left(1+2\alpha_h(t_{ih}+\Delta t_h-t_{\mathrm{ref}})\right)}\right.$$
$$\left.-\frac{\left(1+2\alpha_h(t_{il}-t_{\mathrm{ref}})\right)}{\left(1+2\alpha_h(t_{il}+\Delta t_l-t_{\mathrm{ref}})\right)}\right]+\left(\rho_{fh}-\rho_{fl}\right)gh$$
$$+\Delta p_{\mathrm{rep}}+\Delta p_{c-\mathrm{frep}}+\Delta p_{c-\mathrm{fsen}}+\Delta p_{\mathrm{tilt}}\ , \qquad (1)$$

where,

p_i	is pressure measured during the initial cross-float and is given by

$$p_i = \frac{m_{ih}\, g\left[1-(\rho_a-1.2)\left(\frac{1}{\rho_m}-\frac{1}{8000}\right)\right]\left(1-\frac{\rho_a}{8000}\right)+\sigma c_h}{A_{0h}(1+\lambda_h p_n)(1+2\alpha_h(t_{ih}-t_{\mathrm{ref}}))} \qquad (2)$$

$\Delta p_n = \frac{\Delta m\, g}{A_{0h}}$	is a nominal differential pressure
$p_n = \frac{m_{ih}\, g}{A_{0h}}$	is nominal pressure measured during the initial cross-float (nominal line pressure)
Δm	is the additional mass of the weights used for generating differential pressure
g	is the local gravity
ρ_a	is the density of ambient air
ρ_m	is the density of the weights
σ	is the surface tension of the working fluid
c_h	is the circumference of the piston
A_{0h}	is the effective area of the piston-cylinder assembly at null pressure and reference temperature t_{ref}
λ_h	is the pressure distortion coefficient
α_h	is the linear thermal expansion coefficient of the piston and cylinder ($\alpha_{ph} = \alpha_{ch} = \alpha_h$)
t_{ref}	is the reference temperature of the piston-cylinder assembly
t_{ih}, t_{il}	are the temperatures of the piston-cylinder assemblies recorded on the high and low side of the system during initial cross-float
$\Delta t_h, \Delta t_l$	are differences between temperatures recorded during the pressure measurement (t_h and t_l) and those recorded during the initial cross-float (t_{ih} and t_{il})
ρ_{fh}, ρ_{fl}	are the densities of working fluid of high and low side in the second step
h	is the difference in altitude between the level where the pressure has to be measured and the reference level of the pressure balance (Figure 1)
m_{ih}	is the total mass of the weights loaded on the 'high' pressure balance to generate the line pressure
Δp_{rep}	measurement repeatability
$\Delta p_{c-\mathrm{frep}}$	cross-float repeatability
$\Delta p_{c-\mathrm{fres}}$	resolution of the cross-float
Δp_{tilt}	tilt of the piston

Subscript h and l in the above equations identify the high and low side of the system on Figure 1, while i refers to the quantities recorded during the initial cross-float.

Air density in Eq. (1) and (2) was calculated based on measuring environment conditions using approximation formula [8] while density of working fluid (di-ethylhexylsebacate-DHS) was calculated according to [9].

4. MONTE CARLO METHOD

The MCM completely depends on the quality of the random number generator and the appropriate software. The random number generator should have properties specified in [1] and should pass the Big Crush tests [10]. In our study we use one of those generators. It is based on the Mersenne Twister algorithm that has accuracy up to 32-bit and 2^{219937}-1 cycle length [11]. The results of numerical calculations provided by the AMCM are grouped into batches of fixed size. The size of the batch is defined in advance (10^4), and for each batch, the results of interest are calculated. The estimate of the differential pressure was calculated as the average of 10^4 model values, and the endpoints were obtained using the statistics of the binomial distribution [1], [12]. After the initial batch, a new batch is formed, and the decision to stop computing is made when the desired quality of the results is reached. Otherwise, another batch is formed, and the decision is taken again. According to GUM S1, measure of the quality of the results is the standard deviation associated with the average of the results of interest multiplied by the factor 2. In the modified GUM S1 procedure standard deviation is multiplied by 1-α/2 quantile of the t-distribution with h-1 degrees of freedom, where h is the number of batches obtained by AMCM [2], [3]. A measure of the quality is then compared with the requested numerical tolerance δ. If a measure is no larger than numerical tolerance, the results have stabilized, and computing is stopped. The numerical tolerance reflects a number of significant decimal digits regarded as meaningful in the standard uncertainty. The numerical tolerance δ was calculated flowing procedure given in [1]. GUM S1 and modified GUM S1 were used to achieve a requested numerical accuracy of the estimate of the differential pressure and the left- and right-hand endpoints at a 95% confidence level. The performance of the GUM S1 and modified GUM S1 adaptive procedures were analyzed by repeating the procedures 10^4 times and determining the frequency distribution of the results of interest. In each run, a different seed for the random generator was used and the number of batches was recorded for each run. Frequency distribution for the number of batches was also determined. An example is given for differential pressure of 100 kPa at 42 MPa line pressure. Table 1 summarizes the input quantities with assigned PDFs. In the table, a Gaussian distribution N(μ, σ^2) is described in terms of expectation μ and standard deviation σ, rectangular R(a, b) and triangular distribution T(a, b) with endpoints a and b in terms of expectation (a + b)/2 and semi-width (b - a)/2. For a given example expectation was $\widehat{\Delta\boldsymbol{p}} = 99.975$ kPa and standard uncertainty $\boldsymbol{u}(\widehat{\Delta\boldsymbol{p}}) = 0.034$ kPa.

5. RESULTS AND DISCUSSION

Frequency distribution for the estimates of the differential pressure and left- and right-hand endpoints were obtained. The success rate for the estimates of the differential pressure for both GUM S1 and the modified GUM S1 adaptive procedure is higher than 95%. The reason for this is the faster convergence of the estimate compared to left- and right-hand endpoints concerning the number of Monte Carlo trials. However, the success rate for left- and right-hand endpoints differ when applying GUM S1 and the modified GUM S1 adaptive procedure (Figure 2). GUM S1 adaptive procedure will provide a success rate below the required 95% (Figure 2a). The interval covering 95% of the results (dashed-dotted lines in figure Figure 2a) is about 50% larger than the requested numerical tolerance interval of width 2δ (δ = 0.0005 kPa). The success rate is slightly below 90%, which is smaller than the required 95%. The reason for the lower success rate is a large number of premature terminations of the calculation (Figure 3a). Applying GUM S1 adaptive procedure for approximately 7% of all runs leads to reaching the stopping condition after the second batch (h = 2). Approximately 83 % of those runs, which is more than 5 % of all runs, will give results outside the requested numerical tolerance interval. Applying the modified GUM S1 improves the success rate to approximately 97% (Figure 2b). The 95 % coverage interval is about 7% smaller than the required numerical tolerance interval. The higher success rate is reached because of simultaneously (not independently) observation of all results of interest. In the case of the modified GUM S1, no noticeable premature termination is observed (Figure 3b). The absence of premature termination is due to application of the t-distribution quantile in AMCM. The success rates applying GUM S1 and modified GUM S1 procedure for the right-hand endpoint is similar as for left-hand endpoint.

6. CONCLUSION

Due to the stochastic nature of the MCM, the numerical quality of the results is questionable. The prescribed numerical accuracy of the coverage interval endpoints is not reached by applying the GUM S1 procedure. A large number of premature terminations of the calculation is the reason for a low success rate. The use of an adequate factor from the t-distribution noticeably improves the performance of the adaptive Monte Carlo procedure and decreases the number of premature terminations of calculation.

Table 1: The input quantities with assigned PDF

Input quantity	Source of uncertainty	PDF	Parameters			
			μ-expectation	***σ-std.dev.***	$\frac{a+b}{2}$	$\frac{b-a}{2}$
Δm	cal. certificate	N	0.05 kg	1.25E-07 kg		
m_{ih}	cal. certificate	N	21 kg	5.25E-05 kg		
g	meas. report	N	9.80600787 m/s^2	1.25E-07 m/s^2		
t_a	cal. certificate	N	- 0.15 ^{o}C	0.05 ^{o}C		
	resolution	R			0 ^{o}C	0.005 ^{o}C
	drift	R			0 ^{o}C	0.10 ^{o}C
	inhomogeneity	R			0 ^{o}C	0.60 ^{o}C
	instability	T			0 ^{o}C	0.50 ^{o}C
	repeatability	N	20.65 ^{o}C	0.033^{o}C		
RH_a	cal. certificate	N	2.20 %	0.25 %		
	resolution	R			0 %	0.005 %
	drift	R			0 %	0.10 %
	inhomogeneity	R			0 %	1.25 %
	instability	T			0 %	6.0 %
	repeatability	N	48.20 %	0.034 %		
p_a	cal. certificate	N	- 12 Pa	3.5 Pa		
	resolution	R			0 Pa	0.005 Pa
	drift	R			0 Pa	0.10 Pa
	inhomogeneity	R			0 Pa	5.0 Pa
	instability	T			0 Pa	15 Pa
	repeatability	N	101062 Pa	3.4 Pa		
ρ_m	tech. spec.	R			7950 kg/m^3	200 kg/m^3
A_{0h}	cal. certificate	N	4.90293E-06 m^2	0.60E-10 m^2		
	drift	R			0 m^2	1.0E-10 m^2
λ_h	cal. certificate	N	3.06E-13 Pa^{-1}	4.50E-14 Pa^{-1}		
$2\ \alpha_h$	tech. spec.	R			4.50E-6 ^{o}C^{-1}	0.86E-6 ^{o}C^{-1}
$t_{ih,}t_h$	cal. certificate	N	0.02 ^{o}C	0.005 ^{o}C		
	resolution	R			0 ^{o}C	0.0005 ^{o}C
	drift	R			0 ^{o}C	0.010 ^{o}C
	instability	T			0 ^{o}C	0.10 ^{o}C
	repeatability	N	21.13/21.23 ^{o}C	0.013 ^{o}C		
$t_{il,}t_l$	cal. certificate	N	0.02 ^{o}C	0.005 ^{o}C		
	resolution	R			0 ^{o}C	0.0005 ^{o}C
	drift	R			0 ^{o}C	0.010 ^{o}C
	instability	T			0 ^{o}C	0.10 ^{o}C
	repeatability	N	21.13/21.21 ^{o}C	0.013 ^{o}C		
σ	tech. spec.	R			0.031 Nm^{-1}	1.55E-3 Nm^{-1}
c_h	cal. certificate	N	7.85E-3 m^2	8.89E-8 m^2		
ρ_{fh}	tech. spec.	R			940.71 kgm^{-3}	9.40 kgm^{-3}
ρ_{fl}	tech. spec.	R			940.57 kgm^{-3}	9.40 kgm^{-3}
h	cal. certificate	N	0.00 m	5.80E-4 m		
p_n	cal. certificate	N	42 MPa	714 Pa		
Δp_n	cal. certificate	N	0.1 MPa	1.7 Pa		
Δp_{rep}	meas. system	N	0 Pa	9.60 Pa		
$\Delta p_{c-\mathrm{frep}}$	meas. system	R			0 Pa	10 Pa
$\Delta p_{c-\mathrm{fres}}$	meas. system	R			0 Pa	10 Pa
Δp_{tilt}	meas. system	R			0 Pa	0.07 Pa

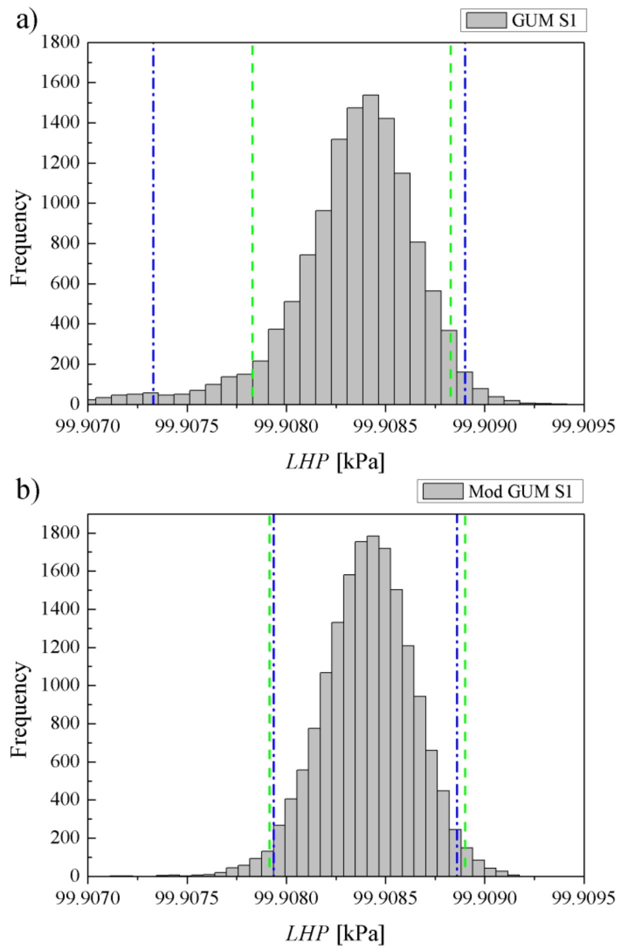

Figure 2: Frequency distribution for the left-hand endpoint obtained from 10^4 runs of GUM S1 and modified GUM S1 adaptive procedure. The dashed lines indicate the endpoints of the requested numerical tolerance interval. The dashed-dotted lines depict an interval containing 95% of the frequency distribution (endpoints of the histogram).

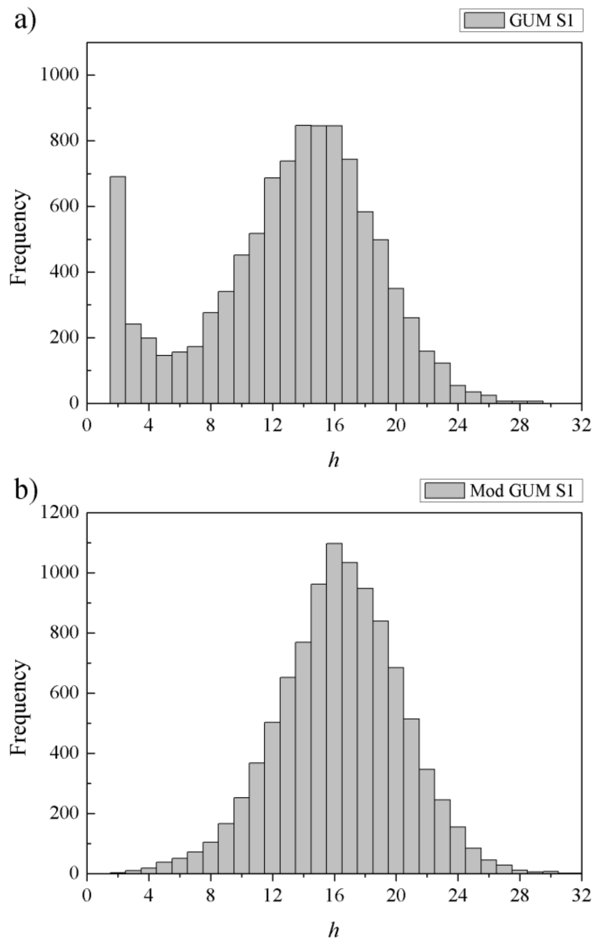

Figure 3: Frequency distribution for the number of batches observed during 10^4 runs of GUM S1 and modified GUM S1 adaptive procedure.

7. REFERENCES

[1] BIMP, IEC, IFCC, ISO, IUPAC, IUPAP, OIML, Evaluation of measurement data — Supplement 1 to the “Guide to the expression of uncertainty in measurement” — Propagation of distributions using a Monte Carlo method, JCGM 101:2008, 2008.

[2] G. Wübbeler, P. M. Harris, M. G. Cox, C. Elster, A two-stage procedure for determining the number of trials in the application of a Monte Carlo method for uncertainty evaluation, Metrologia, 47, 2010, pp. 317–324. DOI: 10.1088/0026-1394/47/3/023

[3] Y. S. Chow, H. Robbins, On the Asymptotic Theory of Fixed Width Sequential Confidence Intervals for the Mean, The Annals of Mathematical Statistics, 36, 1965, pp. 457–462. DOI: 10.2307/2238151

[4] R. S. Dadson, J. E. Daborn, Differential pressure measurement, Physics Bulletin, 24, 1973, pp. 603–604. DOI: 10.1088/0031-9112/24/10/016

[5] J. E. Daborn, The Accurate Measurement of Differential Pressure at High Line Pressure, Measurement and Control 10, 1977, pp. 377–387. DOI: 10.1177/002029407701001003

[6] C. M. Sutton, The Accurate Generation of Small Gauge Pressures Using Twin Pressure Balances, Metrologia, 23, 1987, pp. 187–195. DOI: 10.1088/0026-1394/23/4/003

[7] M. Hay, D. Simpson, Development of high-line differential pressure standards, NPL Report CMAM 41, National Physical Laboratory, 1999.

[8] R. Schwartz, M. Borys, F. Scholz, Guide to Mass Determination with High Accuracy, PTB, 2007.

[9] G. Buonanno, G. Ficco, G. Giovinco, G. Molinar, Ten years of experience in modelling pressure balances in liquid media up to few GPa, Universita degli Studi di Cassino, 2007.

[10] B. A. Wichmann, I. D. Hill, Generating good pseudo-random numbers, Computational Statistics & Data Analysis, 51, 2006, pp. 1614-1622. DOI: 10.1016/j.csda.2006.05.019

[11] M. Matsumoto, T. Nishimura, Mersenne Twister: A 623 Dimensionally Equidistributed Uniform Pseudo-Random Number Generator, ACM Trans. Model. Comput. Simul., 8, 1998, pp. 3-30. DOI: 10.1145/272991.2729953

[12] David, H. A. Order Statistics. Wiley, New York, 1981.