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Some Aspects Concerning the Amplitude Estimation of a Sine Wave by Energy-Based Method

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Abstract-In this paper the influence of quantization noise on the amplitude estimation of a sine wave by energy-based method is investigated. A condition for the number of acquired samples is derived to ensure that the absolute amplitude error due to the quantization noise is smaller than a desired value with a high confidence level. Based on this condition some important conclusions are drawn. Carried out simulation confirms the validity of the derived condition.

I. Introduction

Many engineering applications require high-accuracy amplitude estimations of a sine wave. For this purpose many methods are used. They can be classified in either time-domain methods (parametric methods) or frequency-domain methods (non-parametric methods) [1]. Compared with the time-domain methods the frequency-domain methods can be much easily implemented due to the availability of Fourier analysis packages and not imply a rigid signal model. Moreover, frequency-domain methods are mostly not sensitive to harmonic and non-harmonic distortions and consequently more accurate than time-domain methods.

One of the frequency-domain methods used to estimate the amplitude of a sine wave in non-coherent sampling mode is the energy-based method. This method provides accurate amplitude estimates and can be very easily implemented [2]–[4].

The influence of noise on the amplitude estimation of a sine wave by energy-based method has been studied in the scientific literature [3], [5]. In [3] a condition for the number of acquired samples is derived in order to accurate estimate the rms value of a multi-frequency signal component. This condition depends on the signal-to-noise ratio corresponding to the multi-frequency signal component and on the normalized standard deviation of the multi-frequency signal component power estimator also derived in [3]. In [5] the standard deviation of the rms value of a sine wave corrupted by quantization noise is derived.

The goal of this paper is to determine, based on the expression derived in [5] a simple condition for the number of acquired samples to ensure that the absolute error of amplitude of a sine wave corrupted by quantization noise estimated by energy-based method is smaller than a desired value with a high confidence level. The validity of the derived condition is then verified by means of computer simulations.

II. Theoretical background

Let us consider a sine wave sampled at f_s frequency:

$$x(m) = A\sin\left(2\pi\frac{f_{in}}{f_s}m + \varphi\right), \qquad m = 0,1,2,\ldots,M \tag{1}$$

where A, f_{in} and φ are, respectively, the amplitude, frequency and phase of the sine wave and M is the number of acquired samples. In order to satisfy the Nyquist criteria, f_{in} is smaller than $f_s/2$.

The relationship between the frequencies f_{in} and f_s is:

$$\frac{f_{in}}{f_s} = \frac{\lambda_0}{M} = \frac{l+\delta}{M}, \tag{2}$$

where λ_0 is the number of recorded sine wave cycles, l and δ are, respectively, the integer part and the fractional

part of λ_0. δ is related to the non-coherent sampling mode and $-0.5 \le \delta < 0.5$.
When $\delta \ne 0$, the sampling process is non-coherent with the input sine wave (non-coherent sampling mode), situation most encountered in practice. In this mode the sine wave spectrum is affected by spectral leakage errors. In order to suppress these errors the well-known windowing approach is used leading to the spectral analysis of $x_w(m) = x(m)\cdot w(m)$, where $w(m)$ is the window sequence [6]. The windows frequently used are the cosine-windows. These are defined as:

$$w(m) = \sum_{h=0}^{H-1} (-1)^h a_h \cos\left(\frac{2\pi hm}{M}\right), \qquad m = 0, 1, \ldots, M-1 \qquad (3)$$

where H is the window's order and a_h are the window's coefficients.
The discrete-time Fourier transform (DTFT) of $x_w(m)$ is given by:

$$X_w(\lambda) = \frac{A}{2j}\left[W(\lambda - \lambda_0)e^{j\varphi} - W(\lambda + \lambda_0)e^{-j\varphi}\right], \qquad \lambda \in [0, M) \qquad (4)$$

where λ represents the normalized frequency expressed in bin and $W(\lambda)$ is the DTFT of $w(m)$. The second term from the square brackets of (4) represents the image part of the spectrum. If $W(.)$ exhibits side lobes with negligible level then in the vicinity of λ_0 the contribution from the image part can be neglected.
The energy-based method estimates the power of the sine wave by means of the power of the spectral components centered on the frequency f_{in} and situated inside a frequency band covering the window spectrum main lobe, which contains $2H + 1$ components. The amplitude A is estimated by the energy-based method as [2]–[4]:

$$\hat{A} = \frac{2}{M}\sqrt{\frac{\sum_{k=l-H}^{l+H} \left|X_w(k)\right|^2}{NNPG}}, \qquad (5)$$

where NNPG is the window normalized noise power gain, given by [7]:

$$NNPG = \frac{\sum_{m=0}^{M-1} w^2(m)}{M} = a_0^2 + 0.5\sum_{h=0}^{H-1} a_h^2. \qquad (6)$$

III. Imposed condition to the number of acquired samples

Due to the digitizing process the quantization noise is always present in sampled signals. The standard deviation of the amplitude of a sine wave corrupted by quantization noise estimated by energy-based method can be approximated by [5]:

$$\sigma_A \cong \sqrt{\frac{2ENBW0}{M}}\,\sigma_q, \qquad (7)$$

where σ_q is the quantization noise standard deviation, equal to:

$$\sigma_q = \frac{q}{\sqrt{12}} = \frac{FSR}{2^n\sqrt{12}}, \qquad (8)$$

in which FSR is the full-scale range of the used analog-to-digital converter (ADC), q is its quantization step and n is its resolution;
ENBW0 is the equivalent noise bandwidth of the squared window, equal to [5]:

$$ENBW0 = \frac{M\sum_{m=0}^{M-1} w^4(m)}{\left(\sum_{m=0}^{M-1} w^2(m)\right)^2}. \qquad (9)$$

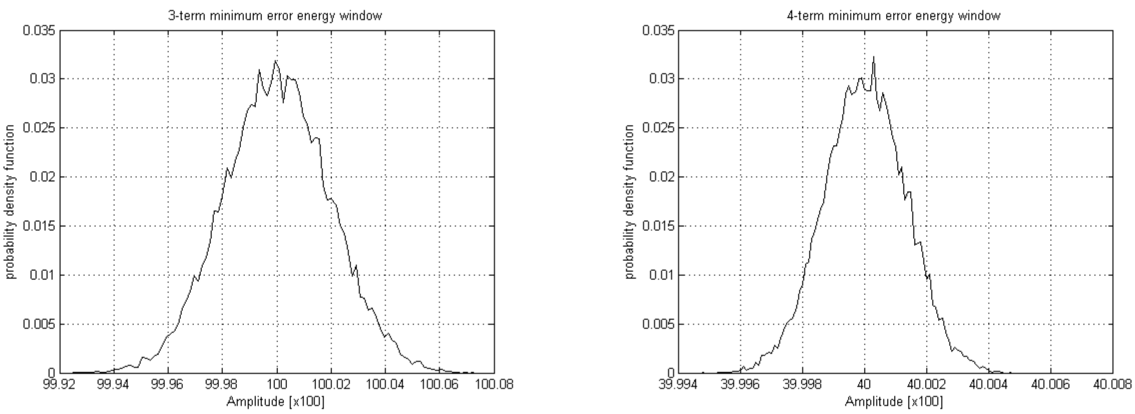
In Table 1 the values of ENBW0 for ones of the most used cosine windows are given [4], [8]–[10].

Table 1. Values of the ENWB0.

Window	Type	ENBW0 (bin)
Maximum side lobe decay	H = 3	2.6265
Rapid side lobe decay 18 dB/oct.		2.4139
Blackman-Harris −71 dB side lobe level		2.3352
Minimum error energy		2.4160
Maximum side lobe decay	H = 4	3.1673
Rapid side lobe decay 30 dB/oct.		2.9220
Blackman-Harris −92 dB side lobe level		2.7632
Minimum error energy		2.8320
Maximum side lobe decay	H = 5	3.6289
Minimum error energy [1]		3.1526
Maximum side lobe decay	H = 6	4.0383
Minimum error energy[2]		3.4197
Maximum side lobe decay	H = 7	4.4100
Blackman-Harris −191dB side lobe level		3.6728

[1] The coefficients of this window are [10]: a_0 = 0.31516086039782, a_1= 0.467, a_2 = 0.18301739393731, a_3 = 0.03299781119189 and a_4 = 0.00182393447297.
[2] The coefficients of this window are [10]: a_0 = 0.29084692392447, a_1 = 0.45, a_2 = 0.20368610117409, a_3 = 0.04983846249154, a_4 = 0.00546711753710 and a_5= $1.613948728039398\cdot10^{-4}$.

It should be pointed out that for a given window's order the minimum error energy windows have the highest effectiveness in the leakage reduction [7]. Moreover, as it can be seen from Table 1 these windows have smaller values of ENWB0 than most other windows.
The quantization noise is usually assumed to be a uniform distributed stationary white noise and statistically independent with the input signal. From these reasons the sine wave amplitude estimates obtained by energy-based method exhibit an approximate normal distribution [11].
Fig. 1 shows the probability density function (pdf) of amplitude estimates for theoretical value set at A = 1 (Fig. 1a) and A = 0.4 (Fig. 1b). The following parameters are used: $\varphi = \pi/3$ rad, l = 259 and M = 4096. The ADC is an ideal one with FSR equal to 5. For A = 1 are used: $\delta = -0.1$, an 8-bit ADC resolution and the 3-term minimum error energy window [7], [9]. For A = 0.4 are used: δ = 0.3, a 12-bit ADC and the 4-term minimum error energy window [7], [9]. It is assumed that the ADC quantization noise is the only noise which affects the sine wave. This noise is modelled by uniformly distributed additive noise. 20,000 trials are used.



(a) A = 1 (b) A = 0.4

Fig. 1. The probability density function of the A estimated values.

Since the estimated value of A has an approximate normal distribution it follows that the modulus of the absolute error of A, $|\Delta A|$ is smaller than $c\sigma_A$, whit a probability done by coverage factor c. In order to obtain a high confidence level c is set to 3:

$$|\Delta A| \le 3\sigma_A. \tag{10}$$

To have an absolute error $|\Delta A|$ smaller than a desired value V_A the following condition must be satisfied:

$$3\sigma_A < V_A. \tag{11}$$

From (7) it can be established that M must satisfy the following condition:

$$M > \frac{18ENBW0}{V_A^2}\sigma_q^2. \tag{12}$$

Let us consider $V_A = kq$, $k > 0$. Thus, from (8) the condition (12) becomes:

$$M > \frac{1.5ENBW0}{k^2}. \tag{13}$$

The above condition is very simple. In order to reduce the computational time M it is recommended to be equal to the nearest integer power of two which satisfy the condition (13).
From the condition (13) some important conclusions can be drawn:
- the derived condition depends on the window used (by means of ENBW0) and on the desired maximum absolute error of A (by means of k);
- the derived condition does not depends on A;
- as the choice of the window depends on the ADC resolution [7], it follows that M depends on the ADC resolution;
- it is sufficient to use a relatively small value of M to estimate larger amplitudes with high accuracy;
- to estimate amplitudes smaller than $4q$ with high accuracy it is necessary to use a large number of samples (for example when the 4-term minimum error energy window is used to estimate an amplitude equal to $2q$ with an relative error smaller than 1 % it is necessarily to have $M > 10620$).

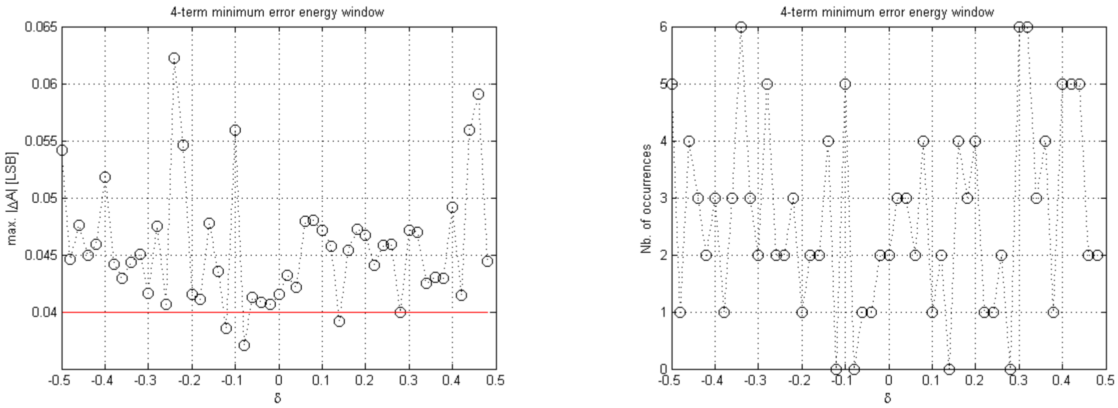
An explanation of the last conclusion is that as M increases the noise floor in the amplitude frequency spectrum decreases and the amplitudes appear much above white noise spread into more frequency bins; so, the amplitudes can be more accurately estimated [12]. In this case because the memory constraints did not allow a single-record approach it is necessarily to collect a number of records with a smaller number of samples than the one given by (13). Then, the amplitude is determined as the mean of the amplitudes estimated for each record.

IV. Computer simulation results

The aim of this section is to verify the effectiveness of the condition (13) by means of the computer simulations. The signal used in simulation is:

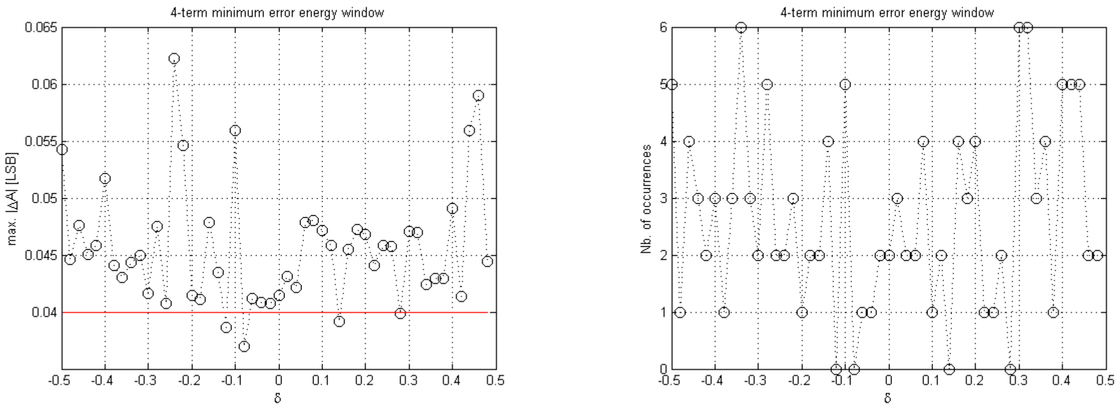
$$x(m) = A\sin\left(2\pi\frac{l+\delta}{M}m + \varphi\right) + e_q(m), \quad m = 0, 1, \ldots, M-1 \tag{14}$$

where $e_q(m)$ is the quantization noise generated by an 12-bit bipolar ADC. The FSR of the ADC is set to 5. The quantization noise is modelled by uniformly distributed additive noise. The 4-term minimum error energy window is used [7]. The error V_A is set to $0.04q$. From (13), $M > 2655$ is obtained. l is set to 237. δ varies in the range $[-0.5, 0.5)$ with an increment of 1/50. For each value of δ, φ is uniformly distributed in the range $[0, 2\pi)$ rad and the maximum of $|\Delta A|$, $|\Delta A|_{max}$ is retained. Each time 1000 runs are done to determine $|\Delta A|_{max}$.
Fig. 2 shows $|\Delta A|_{max}$ as a function of δ (Fig. 2a) and the number of occurrences of absolute errors $|\Delta A|$ higher than V_A as a function of δ (Fig. 2b) for $A = 100q$. M is set to 2656. Fig. 3 shows the same dependencies as in Fig. 2, but for $A = 10q$.
It is evident that the confidence levels obtained for $A = 100q$ are practically the same with the ones obtained for $A = 10q$.
From Figs. 2a and 3a it follows that in the worst case the probability to have all the absolute errors $|\Delta A|$ smaller than V_A is equal to 99.4 %, which is close to the one corresponding to the ideal normal distribution (99.73 %). Thus, the probability to have, for all values of δ, absolute errors $|\Delta A|$ smaller than V_A is higher than 99.4 %.
For $M = 4096$ the probability to have, for all values of δ, absolute errors $|\Delta A|$ smaller than V_A is higher than 99.8% for $A = 10q$ (see Fig. 4). The same confidence levels are, practical obtained, also for $A = 100$q. Compared with the previous case, in this case a higher probability is obtained because from (13) for $M = 4096$, the absolute error $|\Delta A|$ is smaller than $0.032q$ ($< 0.040q$) with a high confidence level.



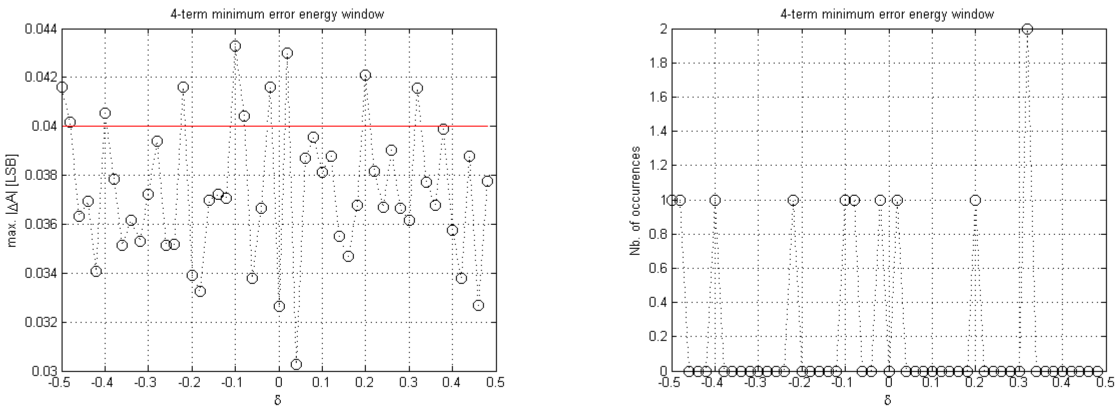
(a) (b)

Fig. 2. (a) $|\Delta A|_{\max}$ as a function of δ; (b) number of occurrences of absolute errors $|\Delta A|$ higher than V_A as a function of δ. A is equal to $100q$ and M is set to 2656.



(a) (b)

Fig. 3. (a) $|\Delta A|_{\max}$ as a function of δ; (b) number of occurrences of absolute errors $|\Delta A|$ higher than V_A as a function of δ. A is equal to $10q$ and M is set to 2656.



(a) (b)

Fig. 4. (a) $|\Delta A|_{\max}$ as a function of δ; (b) number of occurrences of absolute errors $|\Delta A|$ higher than V_A as a function of δ. A is equal to $10q$ and M is set to 4096.

V. Conclusion

In this paper a condition for the number of acquired samples is derived to ensure that the amplitude of a sine wave corrupted by quantization noise is estimated by energy-based method with an absolute error smaller than a desired value with a high confidence level. This condition is very simple and it depends only on the window used (by means of ENBW0 parameter) and on the desired maximum absolute error of amplitude. The validity of the derived condition has been proven by means of computer simulations.

Based on the derived condition it follows that relatively low number of samples are sufficient for estimating larger amplitudes with high accuracy. However, the main drawback of the energy-based method is the fact that large number of samples is required for estimating small sine wave amplitudes with high accuracy.

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