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Estimation of Parameters of the Exponentially Damped Sinusoidal Signals in the Frequency Domain

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Abstract-In this paper, the exponentially damped sinusoidal signals are analyzed. Simple algorithms for fast measurement and estimation of the unknown damping, frequency, amplitude and phase are presented. The peaks of the discrete Fourier transform (DFT) results are adopted to obtain the parameters. The idea of the quotient interpolation using the Hanning window for the basic three parameters can be adopted also for the damping. The Hanning window very efficiently reduce the systematic error in the case of damping estimation, but price for this is in increasing of the estimatation standard deviation in comparison with the rectangular window.

I. Introduction

The estimation of parameters of the exponentially damped sinusoids is of great importance in many applications including in linear system identification, in speech analysis, and in transient analysis. The results of linear and time-invariant differential equations are mostly composed of exponential signal forms. Parameters of exponential signals include frequency, amplitude, damping and phase. There are two types of exponential signals. If the damping is equal zero, the signal component is periodic, conversely, if the damping is not equal to zero, the signal component will be aperiodic, and it will decay to zero with time. Several papers address the problem of estimating the damping besides the basic three signal parameters [1]- [4]. Measurement methods for analyzing and estimation the parameters of the components can be classified as parametric [5],[6] and nonparametric ones [7]-[10]. Parametric methods are model-based and have very good selectivity and statistical efficiency, but require computationally intensive algorithms and very good 'model agreement' with a real multi-component signal. The non-parametric approach to spectrum analysis is computationally efficient and has less sensitivity to algorithm design parameters but has inherent performance limitations as frequency resolution and leakage effects in the non-coherent sampling. This paper presents effective algorithms for fast measurement and estimation of the component damping besides the frequency, amplitude, and phase by the interpolation of the discrete Fourier transform.

II. Estimation of parameters

The sampled multi-component analog signal $g(t)=\sum_m A_m \cdot \mathrm{e}^{-\alpha_m t} \cdot \sin(2\pi f_m t+\varphi_m)$ is obtained in two measurement steps. The data cord is extracted from original signal by sampling ($f_s = 1/\Delta t$ - sampling frequency) and windowing ($w(k)$ - windowing coefficients; $k = 0,1,..., N-1$).

$$g(k\Delta t)_N = w(k)\cdot g(k\Delta t)_\infty = w(k)\cdot \sum_m A_m \cdot \mathrm{e}^{-\alpha_m k\Delta t} \cdot \sin[2\pi f_m k\Delta t+\varphi_m] \tag{1}$$

In the case of the zero damping $\alpha_m = 0$, the DFT at the spectral line i is given by

$$G(i) = -\frac{j}{2}\sum_m A_m\left[W(i-\theta_m)e^{j\varphi_m} - W(i+\theta_m)e^{-j\varphi_m}\right]; \quad \theta_m = \frac{f_m}{\Delta f} = i_m + \delta_m; \quad -0.5 < \delta_m \le 0.5 \tag{2}$$

where $W(*)$ is a spectrum of the window function $w(k)$ and θ_m is a frequency divided by the frequency resolution $\Delta f = 1/(N\Delta t)$. The displacement term δ_m is due to the non-coherent sampling. Non-coherency can be reduced by the interpolation of the DFT coefficients. It has been shown [8] that the best estimation results in reducing long leakage effects gives the three-point estimation using the Hanning window. In the estimation of the particular component m, the three largest local DFT coefficients $\left|G(i_m-1)\right|$, $\left|G(i_m)\right|$, and $\left|G(i_m+1)\right|$ are used for frequency:

$$ {}_3\delta_m \cong 2\frac{\left|G(i_m+1)\right|-\left|G(i_m-1)\right|}{\left|G(i_m-1)\right|+2\left|G(i_m)\right|+\left|G(i_m+1)\right|}, \tag{3} $$

When the displacement δ_m for the specific component is determined, it is possible to estimate also the amplitude by summing the largest three local DFT coefficients around the signal component in the same manner as in the frequency estimation [11]:

$$ {}_3A_m \cong \frac{\pi\delta_m}{\sin(\pi\delta_m)}\frac{\left(1-\delta_m^2\right)\left(4-\delta_m^2\right)}{3}\cdot\left[\left|G(i_m-1)\right|+2\left|G(i_m)\right|+\left|G(i_m+1)\right|\right] \tag{4} $$

and the phase [12]:

$$ {}_3\varphi_m = \frac{(1-\delta_m)\varphi_{i_m-1}+4\varphi_{i_m}+(1+\delta_m)\varphi_{i_m+1}}{6}-\frac{2a\delta_m}{3}+\frac{\pi}{2} \tag{5} $$

where phase parts of the DFT $\varphi_{i_m+s}=\arg\left[G(i_m+s)\right]$ and $s=-1,0,1$ are used.

III. Damping estimation

Estimating of the damping means analyzing of the exponent function $x(t)=\mathrm{e}^{-\alpha t}\cdot u(t)$, where $u(t)$ is the unit step function. The Fourier transform of this expression is [13]:

$$ X(f)=\int_0^{\infty}\mathrm{e}^{-\alpha t}\mathrm{e}^{-\mathrm{j}2\pi ft}\mathrm{d}t=\frac{1}{-(\alpha+\mathrm{j}2\pi f)}\mathrm{e}^{-(\alpha+\mathrm{j}2\pi f)t}\Big|_0^{\infty}=\frac{1}{(\alpha+\mathrm{j}2\pi f)} \tag{6} $$

with amplitude and phase part:

$$ \left|X(f)\right|=\frac{1}{\sqrt{\alpha^2+(2\pi f)^2}};\qquad \arg(X(f))=-\mathrm{arc}\tan\left(\frac{2\pi f}{\alpha}\right) \tag{7} $$

Observation in the limited measurement time T_{M} ($w(t)_{\mathrm{rect}}\cdot x(t)=\mathrm{e}^{-\alpha t}\big|_{0->T_{\mathrm{M}}}$) gives us the following expression:

$$ X(f)=\int_0^{T_{\mathrm{M}}}\mathrm{e}^{-\alpha t}\mathrm{e}^{-\mathrm{j}2\pi ft}\mathrm{d}t=\frac{1}{-(\alpha+\mathrm{j}2\pi f)}\mathrm{e}^{-(\alpha+\mathrm{j}2\pi f)t}\Big|_0^{T_{\mathrm{M}}}=\frac{1}{(\alpha+\mathrm{j}2\pi f)}\left(1-\mathrm{e}^{-(\alpha+\mathrm{j}2\pi f)T_{\mathrm{M}}}\right) \tag{8} $$

$$ X(f)=\frac{T_{\mathrm{M}}}{(\alpha T_{\mathrm{M}}+\mathrm{j}2\pi fT_{\mathrm{M}})}\left(1-\mathrm{e}^{-\alpha T_{\mathrm{M}}}\cdot\mathrm{e}^{-\mathrm{j}2\pi fT_{\mathrm{M}}}\right)=\frac{T_{\mathrm{M}}}{(\theta_\alpha+\mathrm{j}2\pi\theta)}\left(1-\mathrm{e}^{-\theta_\alpha}\cdot\mathrm{e}^{-\mathrm{j}2\pi\theta}\right) \tag{9} $$

where $\theta_\alpha=\alpha T_{\mathrm{M}}$ is the normalized damping like frequency $\theta=f\,T_{\mathrm{M}}$. If the measurement time is long the amplitude part can be approximated by $\left|X(\theta)\right|\approx T_{\mathrm{M}}\Big/\sqrt{\theta_\alpha^2+(2\pi\theta)^2}$, and for the two largest local DFT coefficients δ_m and $1-\delta_m$ apart from the component origin, it can be expressed:

$$ \left|G(i_m)\right|=\frac{T_{\mathrm{M}}}{\sqrt{\theta_\alpha^2+(2\pi\delta_m)^2}},\quad \left|G(i_m+1)\right|=\frac{T_{\mathrm{M}}}{\sqrt{\theta_\alpha^2+(2\pi(1-\delta_m))^2}} \tag{10} $$

Using quotient like in DFT interpolation scheme

$$ \frac{\left|G(i_m+1)\right|}{\left|G(i_m)\right|}=\frac{\sqrt{\theta_\alpha^2+(2\pi(\delta_m))^2}}{\sqrt{\theta_\alpha^2+(2\pi(1-\delta_m))^2}} \tag{11} $$

the expression for θ_α can be deduced:

$$\theta_\alpha = 2\pi\sqrt{\frac{(1-\delta_m)^2|G(i_m+1)|^2-(\delta_m)^2|G(i_m)|^2}{|G(i_m)|^2-|G(i_m+1)|^2}} \tag{12}$$

If θ_α is small $\theta_\alpha \ll 1$ then quotient (11) can be reduced to $Q = |G(i_m+1)|/|G(i_m)| = \delta_m/(1-\delta_m)$ and we get the expression using the rectangular window $w(t)_{\text{rect}}\,\mathrm{e}^{-\alpha t}\big|_{0\to T_\mathrm{M}} \approx w(t)_{\text{rect}}$, for which the spectrum behaviour is well known $|W_{\text{rect}}(\theta)|_{N\gg1} = \sin(\pi\theta)/\pi\theta$ and quotient of the two largest local DFT coefficients can be expressed as:

$$Q_{\text{rect}} = \frac{|G(i_m+1)|}{|G(i_m)|} = \frac{|W_{\text{rect}}(1-\delta_m)|}{|W_{\text{rect}}(\delta_m)|} = \frac{\pi\delta_m}{\sin(\pi\delta_m)}\frac{\sin(\pi(1-\delta_m))}{\pi(1-\delta_m)} = \frac{\delta_m}{1-\delta_m} \tag{13}$$

Using the Hanning window $|W_{\text{hann}}(\theta)|_{N\gg1} = \sin(\pi\theta)/\left(2\pi\theta\left(1-\theta^2\right)\right)$ and adopted conclusions for the rectangular window in the case of $\theta_\alpha \ll 1$, the quotient can be expressed:

$$\begin{aligned} Q_{\text{hann}} &= \frac{|G(i_m+1)|}{|G(i_m)|} = \frac{|W_{\text{hann}}(1-\delta_m)|}{|W_{\text{hann}}(\delta_m)|} = \frac{2\pi\delta_m\left(1-\delta_m^2\right)}{\sin(\pi\delta_m)}\frac{\sin(\pi(1-\delta_m))}{2\pi(1-\delta_m)\left(1-(1-\delta_m)^2\right)} = \\ &= \frac{\delta_m\left(1-\delta_m^2\right)}{(1-\delta_m)\left(1-(1-\delta_m)^2\right)} = \frac{1-\delta_m^2}{(1-\delta_m)(2-\delta_m)} = \frac{1+\delta_m}{2-\delta_m} \end{aligned} \tag{14}$$

This expression for quotient can be used in (11) instead of the expression for the rectangular window and estimation for the damping using the Hanning window can be deduced:

$$\frac{|G(i_m+1)|}{|G(i_m)|} = \frac{\sqrt{\theta_\alpha^2+(2\pi(1+\delta_m))^2}}{\sqrt{\theta_\alpha^2+(2\pi(2-\delta_m))^2}} \quad \rightarrow \quad \theta_{\alpha,\text{hann}} = 2\pi\sqrt{\frac{(2-\delta_m)^2|G(i_m+1)|^2-(1+\delta_m)^2|G(i_m)|^2}{|G(i_m)|^2-|G(i_m+1)|^2}} \tag{15}$$

IV. Evaluation of the estimation algorithms

In simulations, estimation of the damping of the one-component signal in the form $g(\alpha) = 1\cdot\mathrm{e}^{-\alpha}\cdot\sin(2\pi\cdot9.3+0)$ was performed first (Fig.1), where the damping $\alpha = \theta_\alpha/T_\mathrm{M}$ has been changed from 0 to $5\cdot10^{-3}$ and the non-coherent condition was simulated with 9.3 cycles of the signal in the measurement interval T_M.

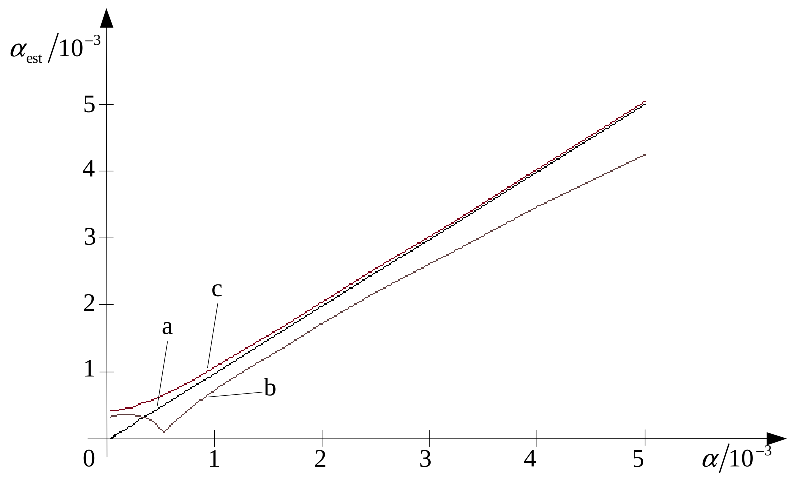

Fig. 1. Estimated values of the damping $\alpha = \theta_\alpha/T_\mathrm{M}$ ($\theta_m = 9.3$, $A_m = 1$, $\varphi_m = 0$): a – reference value of α; b – estimated values using the rectangular window (12), c - estimated values using the Hanning window (15)

The effectiveness of the proposed estimation algorithm using the Hanning window is much better than using the rectangular window due to the faster decreasing of the leakage spectrum tails (Fig .2).

To illustrate accuracy of the proposed algorithms a two component signal (16) with the DFT spectrum (Fig. 2) was used as an example.

$$g(t) = 1.0 \sin\left(2\pi \cdot 9.3t - \frac{\pi}{6}\right) + 1.5e^{-0.5t} \sin\left(2\pi \cdot 20.7t + \frac{\pi}{12}\right) \qquad (16)$$

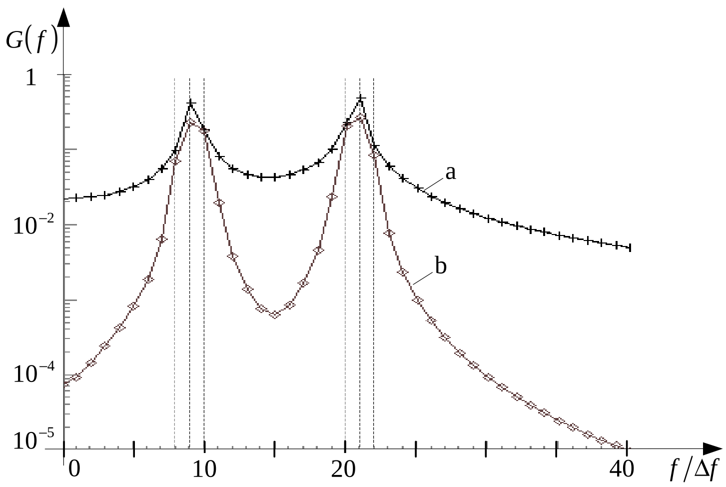

Fig. 2. Spectra of the two-component signal (16); a – using the rectangular window, b – using the Hanning window ($T_M = 1\text{s}$, $N = 1024$)

In the estimations of the component parameters, the largest local DFT coefficients are used (Fig. 2: at the vertical doted lines).
Emulating the real circumstances, random noise with the rectangular distribution and the standard devitation σ_t was added to the signal. The noise propagation through the alghorithms was tested with three values of σ_t (0.001, 0.01, 0.1). The largest standard deviation has the damping estimation (Table 1.). The Hanning window very efficiently reduce the systematic error in the case of damping estimation by (15) $e_{\text{Hann}}(\alpha) = \text{-}15.91 \cdot 10^{-3}$ $\leftrightarrow$ $e_{\text{rect}}(\alpha) = \text{-}509.05 \cdot 10^{-3}$, but price for this is in increasing of the estimatation standard deviation using the rectangular window: $58.89/40.41 \approx 1.46$.

Table 1. Results of the parameters estimations (10000 iterations; $E = (\text{est. value - real value})$, $e = E/\text{real value}$)

parameters		real value	estimated mean value	systematic error	σ ($\sigma_t = 0.001$)	σ ($\sigma_t = 0.01$)	σ ($\sigma_t = 0.1$)
f_1		9.3	9.2996	$E = \text{-}0.408 \cdot 10^{-3}$	$0.521 \cdot 10^{-4}$	$0.521 \cdot 10^{-3}$	$5.22 \cdot 10^{-3}$
α_1		0	-	-	-	-	-
A_1		1.0	0.9994	$e = \text{-}0.553 \cdot 10^{-3}$	$0.603 \cdot 10^{-4}$	$0.603 \cdot 10^{-3}$	$5.99 \cdot 10^{-3}$
φ_1		-0.5236	-0.5223	$E = 1.281 \cdot 10^{-3}$	$1.77 \cdot 10^{-4}$	$1.77 \cdot 10^{-3}$	$17.7 \cdot 10^{-3}$
f_2		20.7	20.7014	$E = 1.396 \cdot 10^{-3}$	$0.450 \cdot 10^{-4}$	$0.450 \cdot 10^{-3}$	$4.39 \cdot 10^{-3}$
α_2	Hann	0.5	0.4920	$e = \text{-}15.91 \cdot 10^{-3}$	$5.855 \cdot 10^{-3}$	$58.89 \cdot 10^{-3}$	$58.89 \cdot 10^{-3}$
	rect		0.2455	$e = \text{-}509.05 \cdot 10^{-3}$	$3.984 \cdot 10^{-3}$	$40.41 \cdot 10^{-3}$	$40.41 \cdot 10^{-3}$
A_2 , $\overline{A}_2$		1.5, 1.1807	1.1734	$e_{\overline{A}_2} = \text{-}6.174 \cdot 10^{-3}$	$0.510 \cdot 10^{-4}$	$0.510 \cdot 10^{-3}$	$5.04 \cdot 10^{-3}$
φ_2		0.2618	0.2824	$E = 20.6 \cdot 10^{-3}$	$1.453 \cdot 10^{-4}$	$1.453 \cdot 10^{-3}$	$14.24 \cdot 10^{-3}$

Since the estimated amplitude by the interpolated DFT is the average value in the measurement interval $T_M = 1\text{s}$, it is convenient to use the mean value $\overline{A}_2 = 1.1807$ of the damped amplitude of the second component in the measurement interval and in this case the relative form of the systematic error is $e_{\bar{A}_2} = -6.174 \cdot 10^{-3}$.

IV. Conclusions

In this paper, simple algorithms for fast measurement and estimation of the unknown damping, frequency, amplitude and phase are presented. Parameters are calculated from the DFT coefficients around the component peaks. The idea of the quotient interpolation using the Hanning window for the basic three parameters can be adopted also for the damping. The Hanning window very efficiently reduce the systematic error in the case of damping estimation, but price for this is in increasing of the estimation standard deviation in comparison with the rectangular window.

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