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Optimal Windows for Sine-Wave Amplitude Estimation by the Energy-Based Method

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Abstract- In this paper a criterion for the choice of the optimal window for the amplitude estimation of a sine-wave corrupted by a stationary white noise using the energy-based method (EBM) is proposed. The window is chosen by ensuring that the spectral leakage effect on the estimator accuracy is negligible with respect to errors due to noise. The proposed criterion is applied to several different settings, and some interesting conclusions are drawn. The effectiveness of the proposed criterion is validated by means of both computer simulations and experimental results.

I. Introduction

In many engineering applications the amplitude of a sine-wave must be known with high accuracy. The adopted estimation procedures can be classified as either time-domain (parametric) or frequency-domain (nonparametric) methods. The latter ones are often used because they are more robust with respect to uncertainties in the signal model and are easily to implement. The main drawback of the frequency-domain methods is the well known leakage effect due to the noncoherent sampling.

A frequency-domain procedure often used to compensate for the leakage effect is the so-called energy-based method (EBM), which was proposed in [1], [2]. This method provides accurate amplitude estimates and can be very easily implemented [1]-[5]. The performance of the EBM strongly depends on the adopted window, which usually belongs to the cosine class [1]-[4]. A criterion for the choice of the optimal window for the amplitude estimation of a multifrequency signal component by the EBM has been proposed in [4]. This is a complex criterion based on several parameters, which must be properly set to achieve accurate estimates.

The aim of this paper is to propose a simpler criterion for the choice of the optimal window for the amplitude estimation of a sine-wave corrupted by a stationary white noise by the EBM. White additive noise is considered since it allows accurate modelling of many phenomena occurring in practical applications, such as signal quantization or wide band superimposed noise.

The proposed criterion has been applied to several different settings. Based on the results achieved some interesting conclusions are drawn. Moreover, the effectiveness of the proposed criterion has been verified by means of both computer simulations and experimental results.

II. Amplitude estimation by the energy-based method

Let us consider a discrete-time sine-wave of amplitude A, frequency f_{in}, and phase φ, sampled at frequency f_s, i.e.

$$x(m) = A\sin\left(2\pi \frac{f_{in}}{f_s} m + \varphi\right), \qquad m = 0,1,2,\ldots,M-1 \tag{1}$$

The frequency f_{in} is chosen smaller than $f_s/2$ to satisfy the Nyquist theorem. When M samples are acquired, the ratio between the frequencies f_{in} and f_s can be expressed as:

$$\frac{f_{in}}{f_s} = \frac{\lambda_0}{M} = \frac{l+\delta}{M}, \tag{2}$$

where l and δ are respectively the integer and the fractional parts of the number of acquired sine-wave cycles λ_0, and $-0.5 \le \delta < 0.5$. If the sine-wave is sampled coherently we have $\delta = 0$. Conversely, noncoherent sampling is characterized by $\delta \neq 0$. The latter case is very common in practical applications. In this case the sine-wave spectrum is affected by the well known leakage effect. In order to reduce the spectral leakage, the signal $x(m)$ is weighted by a suitable window, $w(m)$, i.e. $x_w(m) = x(m) \cdot w(m)$. Windows adopted in the EBM belong to the well known cosine class windows [1], [2], [6]:

$$w(m) = \sum_{h=0}^{H-1} (-1)^h a_h \cos\left(\frac{2\pi hm}{M}\right), \qquad m = 0,1,\ldots,M-1 \tag{3}$$

where H represents the number of window terms and a_h are the window coefficients.
The Discrete-Time Fourier Transform (DTFT) of the signal $x_w(\cdot)$ is given by:

$$X_w(\lambda) = \frac{A}{2j}\left[W(\lambda - \lambda_0)e^{j\varphi} - W(\lambda + \lambda_0)e^{-j\varphi}\right] \qquad \lambda \in [0, M) \tag{4}$$

where λ is the continuous frequency expressed in bins and $W(\cdot)$ is the DTFT of the window $w(\cdot)$. For $|\lambda| << M$, this is can be expressed as [6]:

$$W(\lambda) = \frac{M\lambda \sin(\pi\lambda)}{\pi} e^{-j\pi\lambda\frac{M-1}{M}} \sum_{h=0}^{H-1} \frac{(-1)^h a_h}{\lambda^2 - h^2}. \tag{5}$$

Notice that the second term in (4) represents the image component of the sine-wave spectrum.
In the EBM the power of the sine-wave is evaluated by using a small number of Discrete Fourier Transform (DFT) samples centred around the spectrum peak, which is located in the frequency bin l [1], [2], [4]. In particular, for the H-term cosine windows it has been shown that the use of $(2H + 1)$ DFT samples is advantageous since it ensures a very good compromise between spectral leakage reduction, selectivity capability of nearby spectral components, and computational effort [4]. Based on this result, the sine-wave amplitude A is estimated as [1]-[4]:

$$\hat{A} = \frac{2}{M}\sqrt{\frac{\sum_{k=l-H}^{l+H} |X_w(k)|^2}{NNPG}}, \tag{6}$$

where $NNPG$ is the window Normalized Noise Power Gain, defined as $NNPG = \sum_{m=0}^{M-1} w^2(m)/M$ [7]. For the H-term cosine windows we have: $NNPG = a_0^2 + 0.5\sum_{h=0}^{H-1} a_h^2$.

For $\delta = 0$ and the windows commonly employed, using (4) and (5), it can shown that $\hat{A} = A$. Thus, when the sine-wave is sampled coherently, its amplitude is exactly estimated. For $\delta \neq 0$, using (4) and (5), it can be shown that $\hat{A} \neq A$. In fact, the amplitude estimator is affected by both the spectral interference from the image component and the bias due to the spectral leakage, called in the following algorithm bias. The spectral interference becomes negligible if the value of l is high enough, which corresponds to a high number of acquired sine-wave cycles. Conversely, the algorithm bias becomes negligible by choosing a suitable window.

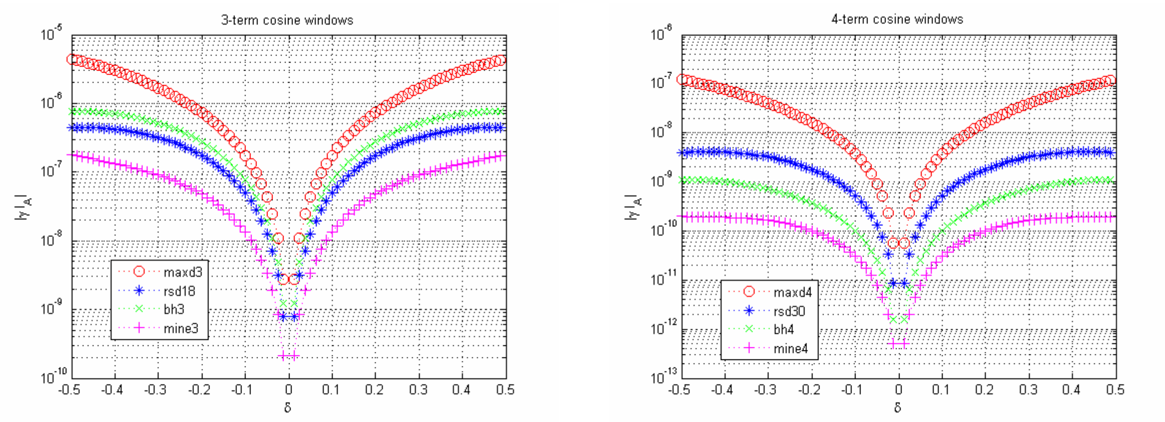
III. Amplitude estimation bias due to the algorithm

Let us assume at first that the integer part l of the number of acquired sine-wave cycles is high enough that the effect of the image component on the estimator $\hat{A}$ is negligible. Thus, from (4) and (5) we obtain:

$$\hat{A} \cong \frac{A}{M}\sqrt{\frac{\sum_{k=-H}^{H} |W(k-\delta)|^2}{NNPG}}. \tag{7}$$

Expression (7) allows us to determine the amplitude relative bias due to the algorithm, γ_A^l, which depends on both the window used and the fractional part δ of the acquired sine-wave cycles. Fig. 1 shows the absolute value of γ_A^l as a function of δ for some commonly used cosine windows [1], [2], [6]:

- 3-term cosine windows (H = 3): maximum sidelobe decay window (*maxd3*), rapid sidelobe decay 18 dB/octave window (*rsd18*), Blackman-Harris window with the highest peak sidelobe equal to −71 dB (*bh3*), and the minimum error energy window (*mine3*);
- 4-term cosine windows (H = 4): maximum sidelobe decay window (*maxd4*), rapid sidelobe decay 30 dB/octave window (*rsd30*), Blackman-Harris window with the highest peak sidelobe equal to −90 dB (*bh4*), and the minimum error energy window (*mine4*).



(a) (b)

Fig. 1. Relative amplitude bias $\left|\gamma_A^l\right|$ due to spectral leakage as a function of δ for commonly used 3-term (a) and the 4-term (b) cosine widows.

As we can see the bias $\left|\gamma_A^l\right|$ decreases as H increases and its maximum $\left|\gamma_A^l\right|_{\max}$ is reached for $\delta = -0.5$. Thus, the maximum magnitude of the estimation bias is given by:

$$\left|\Delta^l A\right|_{\max} \cong A\left(1-\frac{1}{M}\sqrt{\frac{\sum_{k=-H}^{H}\left|W(k-0.5)\right|^2}{NNPG}}\right) = A\left|\gamma_A^l\right|_{\max}. \tag{8}$$

As expected, the minimum error energy windows exhibit the best performance since they have been designed by minimizing the window energy outside a frequency band centred around the spectrum peak and width equal to $(2H + 1)$ DFT samples.

IV. The proposed window selection criterion

In order to model common real-life situations we assume that a stationary white noise with zero mean and variance σ_n^2 is added to the sine-wave. In this case the amplitude estimator $\hat{A}$ exhibits an almost normal distribution [8], [9]. Thus, the amplitude estimation error due to noise satisfies with high probability the following constraint:

$$\left|\Delta^n A\right| \le c\cdot\sigma_{\hat{A}}, \tag{9}$$

where c is the suitable coverage factor (i.e. c = 3) and $\sigma_{\hat{A}}$ is the standard deviation of the amplitude estimator $\hat{A}$ given by [4]:

$$\sigma_{\hat{A}} \cong \sqrt{\frac{2ENBW0}{M}}\,\sigma_n, \tag{10}$$

where $ENBW0$ is the Equivalent Noise BandWidth of the squared window, defined by $ENBW0 = M\sum_{m=0}^{M-1} w^4(m) / \left(\sum_{m=0}^{M-1} w^2(m)\right)^2$ [4].

The window effectiveness with respect to the spectral leakage reduction is high when the maximum bias (8) is much smaller than the estimation error (9), that is:

$$\left|\Delta^l A\right|_{\max} \le \frac{\left|\Delta^n A\right|_{\max}}{\mu}, \tag{11}$$

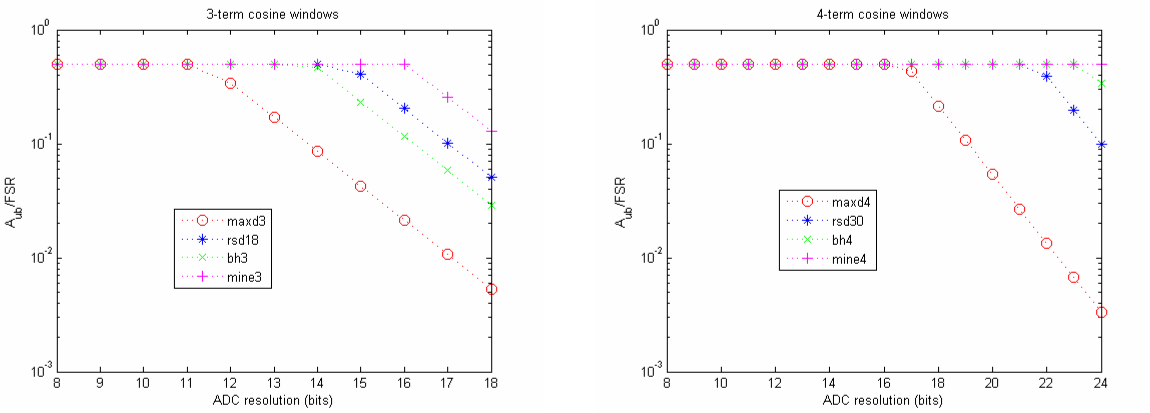
where μ is a positive, sufficiently high, real number. In the following of the paper we will assume $\mu = 10$.
From (8), (9), and (10), the following constraint for the amplitude A can be derived:

$$A \le \frac{c\sqrt{\frac{2ENBW0}{M}}}{\mu\left|\gamma_A^l\right|_{\max}}\sigma_n. \tag{12}$$

Since the noise power is always higher than or equal to the quantization noise power, a more strict constraint can be achieved by substituting σ_n with the quantization noise standard deviation σ_q. If an n-bit Analog-to-Digital Converter (ADC) with Full Scale Range FSR is used, we have $\sigma_q = FSR/\left(2^n\sqrt{12}\right)$ Thus, the constraint (11) is surely satisfied if the sine-wave amplitude is less than the following upper-bound:

$$A_{ub} = FSR\frac{c\sqrt{\frac{2ENBW0}{M}}}{\mu 2^n\sqrt{12}\left|\gamma_A^l\right|_{\max}}. \tag{13}$$

Obviously, if the value returned by (13) is higher than $FSR/2$, then the sine-wave amplitude is limited by $FSR/2$.
Expression (13) implies that for a given window and FSR, A_{ub} decreases when M and/or n increases.
Clearly, the spectral leakage suppression capability of the window increases as A_{ub} increases. Therefore (13) represents a criterion for the selection of the optimal window for amplitude estimation using the EBM.
In particular, it should be noticed that if the amplitude A to be estimated satisfy the upper bound returned by (13) for two or more different windows, then the window providing the minimum estimator variance $\sigma_{\hat{A}}^2$ should be adopted. In fact, in this case the highest estimator repeatability is achieved.
Fig. 2 shows the ratio A_{ub}/FSR as a function of the ADC resolution when the above defined 3- and 4-term cosine windows are adopted. The number of samples M was equal to 1024, and the ADC FSR was equal to 10. As we can see, the 3-term cosine windows suffice when the used ADCs have small and moderate resolutions. Conversely, the 4-term cosine windows must be adopted with high resolution ADCs. As expected, the minimum error energy windows exhibit the best performance. Opposite, the maximum sidelobe decay windows have the poorest spectral leakage suppression capability.



(a) (b)

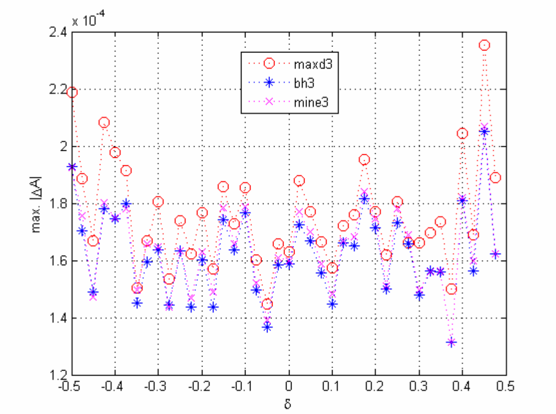
Fig. 2. Ratio A_{ub}/FSR versus ADC resolution when the 3-term (a) and 4-term (b) cosine windows are adopted.

V. Simulation and experimental results

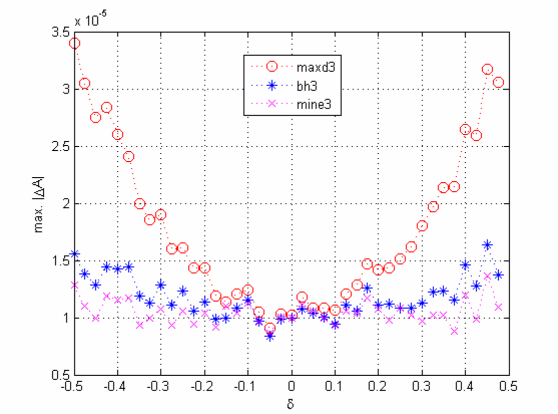
The aim of this section is to validate the proposed criterion by means of computer simulations and experimental results.

In the simulations runs, a pure sine-wave was digitized using an ideal bipolar ADC with *FSR* equal to 10. The amplitude A was set to 4.9, the integer part l was set to 93, and the number of acquired samples M was set to 1024. The quantization noise was modelled as a uniformly distributed additive noise. The fractional part of the acquired sine-wave cycles δ was varied in the range [−0.5, 0.5) with a step of 1/40. For each value of δ, the sine-wave phase φ was chosen at random in the range [0, 2π) rad and the maximum absolute value of the amplitude error $|\Delta A|_{max}$ was determined.

Fig. 3 shows the achieved estimation error $|\Delta A|_{max}$ as a function of δ when the 3-term maximum sidelobe decay, Blackman-Harris, and minimum error energy windows are adopted and the ADC resolution was 12 bits (Fig. 3(a)) and 16 bits (Fig. 3(b)) respectively.



(a)



(b)

Fig. 3. Maximum amplitude estimation errors $|\Delta A|_{max}$ returned by simulations as a function of δ. The 3-term maximum sidelobe decay, Blackman-Harris, and minimum error energy windows are considered. The ADC resolution is 12 bits (a) and to 16 bits (b).

According to the proposed criterion (see Fig. 2(a)), when the ADC resolution is 12 bits, the amplitude estimates exhibit almost the same accuracy when the minimum error energy window and the Blackman-Harris window are adopted and somewhat smaller accuracy when the maximum sidelobe decay window is used. Conversely, for an ADC resolution of 16 bits, the best estimator accuracy is achieved using the minimum error energy window. Moreover, the Blackman-Harris window exhibits almost the same accuracy, while the results provided by the maximum sidelobe decay window are significantly biased by the algorithm. These conclusions are confirmed by the simulation results reported in Fig. 3.

In the experimental setup the sine-waves were supplied by the Agilent 33220A signal generator. The amplitude was set to 4.9 V and the related frequencies were equal to about 9.1 kHz in order to obtain different values of δ in the range (−0.5, 0.5). The signals were acquired with a 12-bit data acquisition board NI-6023E, developed by National Instruments. The *FSR* and the sampling frequency were set 10 V and 100 kHz respectively. For each frequency 500 runs of M = 1024 samples each were acquired and the maximum estimation error $|\Delta A|_{max}$ was computed. At each signal frequency the mean value of the amplitude estimate returned by the four-parameter sine-fit algorithm was used as a reference because of its very high accuracy [10]. The maximum estimation error $|\Delta A|_{max}$ as a function of δ is depicted in Fig. 4.

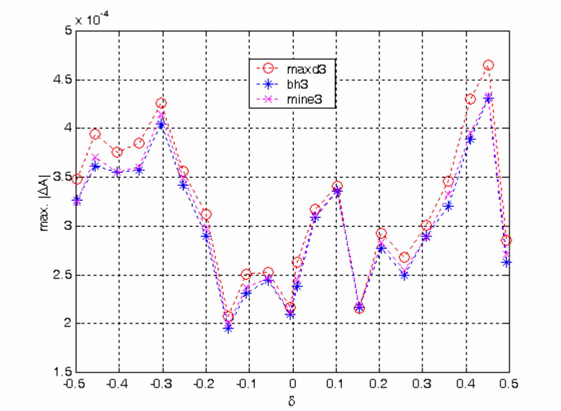


Fig. 4. Maximum amplitude estimation errors $|\Delta A|_{max}$ returned by the experimental results as a function of δ. The 3-term maximum sidelobe decay, Blackman-Harris, and minimum error energy windows are considered. The ADC resolution is 12 bits.

It should be noticed that the acquisition board had about 11 effective bits. Thus, the power of the noise superimposed to the generated sine-wave is quite close to the one introduced by a 12-bit ideal quantizer. For this reason, even though the estimation errors are somewhat higher than those achieved by simulation, the same conclusions can be drawn also from the experimental results. In particular, when the minimum error energy window and the Blackman-Harris window are adopted, the amplitude estimates exhibit almost the same accuracy. Conversely, when the maximum sidelobe decay window is used the estimation accuracy worsen.

VI. Conclusion

A criterion for the choice of the optimal window for the amplitude estimation of a sine-wave corrupted by a stationary white noise using the energy-based method has been proposed. The application of this criterion is quite simple. In particular, it has been shown that the use of suitable 3-term cosine windows allows the achievement of accurate amplitude estimates when acquisition systems with small or moderate resolutions are used. Conversely, when high resolution acquisition systems are employed, the 4-term cosine windows should be adopted. Among the analyzed cosine windows, the 3- and 4-term minimum error energy windows provide the best performance.

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