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Estimation of the Amplitude Quotient of Signals with Common Frequency

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Abstract **-** The paper presents simple single-step algorithms for fast measurement and estimation of the amplitude ratio of the sinusoidal signals with the same frequency from two channels. The idea of the systematic error reduction with the weighted amplitude DFT coefficients around the component peaks is used in the comparative measurement procedure. Paper analyses and compares the systematic bias errors and noise error behaviors of the amplitude quotient estimation changing the order of Rife-Vincent windows class I, which are designed for maximization of the window spectrum side-lobes fall-off. The minimal bias errors are shown in relation to the number of signal cycles in the measurement interval.

Keywords **-** Amplitude quotient, nonparametric estimation, frequency domain, comparative method, Rife-Vincent windows

I. Introduction

In the two channel acquisition systems, it is often to estimate the parameters of two common frequency sinusoidal signals (like in very wide areas of impedance measurements [1]-[2], power measurements [3], etc.). In the comparative measurements (impedance measurement, error evaluation of transformers [4], etc.) values of the amplitude and phase of each channel are not required. The only values needed are in fact the amplitude ratio A_{m2}/A_{m1} and the phase difference $\varphi_{m1}-\varphi_{m2}$ of the investigated components from channel 1 and 2 with the same frequency f_m.

Sampling by frequency $f_s = 1/\Delta t$ of the periodic band limited analog signal $g(t)$ composed of M components (with f_m, A_m, and φ_m as frequency, amplitude, and phase, respectively) can be express as follows:

$$w(n\Delta t)\big|_{n=0,\ldots,N-1} \cdot g(n\Delta t) = w(n\Delta t)\cdot \sum_{m=0}^{M-1} A_m \sin(2\pi f_m n\Delta t + \varphi_m) \tag{1}$$

To estimate parameters of the time dependent signals, containing any periodicity, it is most suitable to use a transformation of the signal in the frequency domain. The discrete Fourier transformation (DFT) of the windowed signal $w(k)\cdot g(k)$ on N sampled points (1) at the spectral line i is given by:

$$G(i) = -\frac{\mathrm{j}}{2}\sum_{m=0}^{M} A_m \left(W(i-\theta_m)\mathrm{e}^{\mathrm{j}\varphi_m} - W(i+\theta_m)\mathrm{e}^{-\mathrm{j}\varphi_m} \right) \tag{2}$$

where $\theta_m = f_m/\Delta f = i_m + \delta_m$ is the component frequency related to base frequency resolution $\Delta f = 1/N\Delta t$ and consists of an integer part and the non-coherent sampling displacement term δ_m.

Evaluations of the time-discrete spectra, however, are hampered by the window leakage, which occurs if a non-integer number of periods are presented in the sampled data set and this is the usual situation in real measurements. The leakage effects can be very reduced with the comparative measurement, if the simultaneousness of the sampling on both channels is assumed and the measurement time of signals is the same. The assurance of these conditions gives equal displacements $\delta_{m1} = \delta_{m2} = \delta_m$. In this paper we try to show the effectiveness of the systematic error reduction using weighted amplitude DFT coefficients of two sine signals with common frequency. In estimations the Rife-Vincent windows class I have been used.

II. Estimation of the amplitude quotient

The DFT coefficients surrounding one signal component are due to the short-range leakage contribution of the window spectrum weighted by the amplitude of the component (from the first term in (2)) and the long-range leakage contributions: from the second term in (2) of the investigated component and from both terms in (2) of other components.

$$|G(i_m)| = \frac{A_m}{2}\left|W(\delta_m)e^{j\varphi_m} - W(2i_m+\delta_m)e^{-j\varphi_m}\right| + \sum_{k=0,k\neq m}^{M}|\Delta(i_k)| \tag{3}$$

For one significant component only, the largest local amplitude DFT coefficient can be expressed with the short-range leakage contribution $A_m|W(\delta_m)|/2$ and the long-range leakage contributions collected in $\Delta(i_m)$ and from here, the amplitude can be deduced:

$$|G(i_m)| = \frac{A_m}{2}|W(\delta_m)| \pm \Delta(i_m) \qquad \Rightarrow \qquad {}_1A_m = 2\frac{|G(i_m)| \mp {}_1\Delta(i_m)}{|W(\delta_m)|} \tag{4}$$

The same expression is valid for amplitudes of both channels ${}_1A_{m,1}$ and ${}_1A_{m,2}$ respectively. Looking for the quotient of the amplitudes of two signals with the same frequency, the window function $|W(\delta_m)|$ in the denominator can be cancelled. Neglecting the long-range leakage contributions at one DFT coefficient ${}_1\Delta(i_m) << |G(i_m)|$, the quotient of the amplitudes can be estimated by the quotient of the maximal local amplitude DFT coefficients of two signals.

$$\frac{{}_1A_{m,1}}{{}_1A_{m,2}} = \frac{|G_1(i_m)| \mp {}_1\Delta_1(i_m)}{|G_2(i_m)| \mp {}_1\Delta_2(i_m)} \qquad \Rightarrow \qquad \frac{{}_1A_{m,1}}{{}_1A_{m,2}} \doteq \frac{|G_1(i_m)|}{|G_2(i_m)|} \tag{5}$$

The long-range leakage contributions can be reduced in three ways: by using windows with faster decreasing of the side lobes than the rectangular window (like Rife-Vincent windows, Dolph-Chebychev windows, etc.), or by using the multi-point interpolated DFT algorithms, or by minimizing the phase difference of the components of the two signals $\varphi_{12} = \varphi_{1,m} - \varphi_{2,m} \to 0$ [5].

In the same manner as in (5), we can obtain the amplitude of the signal by summing the largest two local DFT coefficients around the signal component (*s* is the sign of displacement [6]). Following the fact that the successive leakages changing their sign owing to the measurement time truncation and consequently having the *sinc* function in the window kernel, the long-range leakages $|\Delta(i_m+s)| - |\Delta(i_m)| = |{}_2\Delta(i_m)| < |{}_1\Delta(i_m)|$ can be reduced by summation of the largest local amplitude DFT coefficients:

$$|G(i_m)| + |G(i_m+s)| = \frac{A_m}{2}|W(\delta_m)| \pm |\Delta(i_m)| + \frac{A_m}{2}|W(1-\delta_m)| \mp |\Delta(i_m+s)| = \frac{A_m}{2}\left[|W(\delta_m)| + |W(1-\delta_m)|\right] \pm {}_2\Delta(i_m), \tag{6}$$

$${}_2A_m = 2\frac{|G(i_m)| + |G(i_m+s)| \mp {}_2\Delta(i_m)}{|W(\delta_m)| + |W(1-\delta_m)|} \tag{7}$$

The denominator in (7), which is defined by the employed window, can be cancelled in the quotient:

$$\frac{{}_2A_{m,1}}{{}_2A_{m,2}} = \frac{|G_1(i_m)| + |G_1(i_m+s)| \mp {}_2\Delta_1(i_m)}{|G_2(i_m)| + |G_2(i_m+s)| \mp {}_2\Delta_2(i_m)} \qquad \Rightarrow \qquad \frac{{}_2A_{m,1}}{{}_2A_{m,2}} \doteq \frac{|G_1(i_m)| + |G_1(i_m+s)|}{|G_2(i_m)| + |G_2(i_m+s)|} \tag{8}$$

The single amplitude can be estimated with the three largest local amplitude coefficients [6]:

$${}_3A_m = 2 \cdot \frac{|G(i_m-1)| + 2|G(i_m)| + |G(i_m+1)| \mp {}_3\Delta(i_m)}{|W(1+\delta_m)| + 2|W(\delta_m)| + |W(1-\delta_m)|} \tag{9}$$

The window dependent denominator and the short-leakage window contributions in numerator $|\Delta(i_m-1)| - 2|\Delta(i_m)| + |\Delta(i_m+1)| = |{}_3\Delta(i_m)| << |{}_1\Delta(i_m)|$ can be canceled in the quotient:

$$\frac{{}_3A_{m,1}}{{}_3A_{m,2}} \doteq \frac{|G_1(i_m-1)| + 2|G_1(i_m)| + |G_2(i_m+1)|}{|G_2(i_m-1)| + 2|G_2(i_m)| + |G_2(i_m+1)|}. \tag{10}$$

We can use the same procedure with the weighted five largest coefficients and interpolation. In the last term of the equation (11) the absolute value of the difference is used when one of the coefficients $|G(i_m-2)|$ or $|G(i_m+2)|$ drops out of the spectrum main lobe ($4\Delta f$ wide in the Hann window case) and gets the negative sign.

$$\frac{{}_5A_{m,1}}{{}_5A_{m,2}} \doteq \frac{6\left|G_1(i_m)\right| + 4\left(\left|G_1(i_m+1)\right| + \left|G_1(i_m-1)\right|\right) + \left|\left|G_1(i_m+2)\right| - \left|G_1(i_m-2)\right|\right|}{6\left|G_2(i_m)\right| + 4\left(\left|G_2(i_m+1)\right| + \left|G_2(i_m-1)\right|\right) + \left|\left|G_2(i_m+2)\right| - \left|G_2(i_m-2)\right|\right|} \tag{11}$$

The same procedure can be used also with the seven largest local amplitude coefficients [6]:

$$\frac{{}_7A_{m,1}}{{}_7A_{m,2}} \doteq 2\cdot\frac{20\left|G_1(i_m)\right| + 15\left(\left|G_1(i_m+1)\right| + \left|G_1(i_m-1)\right|\right) + 6\left|\left|G_1(i_m+2)\right| - \left|G_1(i_m-2)\right|\right| - \left|\left|G_1(i_m+3)\right| - \left|G_1(i_m-3)\right|\right|}{20\left|G_2(i_m)\right| + 15\left(\left|G_2(i_m+1)\right| + \left|G_2(i_m-1)\right|\right) + 6\left|\left|G_2(i_m+2)\right| - \left|G_2(i_m-2)\right|\right| - \left|\left|G_2(i_m+3)\right| - \left|G_2(i_m-3)\right|\right|} \tag{12}$$

Fixing the time, the reduction of the systematic leakage error mostly depends on the used window. For the sake of analytical simplicity cosine-class windows are frequently used [7]. Windows of the class I (RV-I) are designed for maximization window spectrum side-lobes fall-off based on a number of the time domain window derivatives zeroes at the window ends [8],[9]:

$$w(n) = \sum_{l=0}^{P} D_{1,l}\cdot\cos\left(l\,\frac{2\pi}{N}\cdot n\right),\quad n = 0, 1, \ldots, N-1 \tag{13}$$

where $D_{1,l}$ are the weighted coefficients of cosines in the window function. A number of the used cosines functions defines window order *P*. When the order *P* is 0, the coefficient $D_{1,0}$ is 1 and the equation (138) gives the rectangular shape. If *P* is 1 we get the Hann window. Higher values of *P* expand the window transform main-lobe and reduce the spectral leakage (Fig. 1). The weighted coefficients can be calculated using expression $C_x^y = x!/((x-y)!\cdot y!)$: $D_{1,0} = \frac{C_{2P}^{P}}{2^{2P}}$; $D_{1,l}(P) = (-1)^l\,\frac{C_{2P}^{P-l}}{2^{2P-1}}$; $l = 1, 2, \ldots, P$.

Frequency spectrum can be generally expressed by [8]:

$$W(\theta) = \frac{(2P)!}{2^{2P}}\cdot\frac{\sin(\pi\theta)}{\pi\theta}\cdot\frac{1}{\prod_{l=1}^{P}\left(l^2-\theta^2\right)}\cdot \mathrm{e}^{-\mathrm{j}\frac{N-1}{N}\theta} = \left|W(\theta)\right|\cdot \mathrm{e}^{-\mathrm{j}\frac{N-1}{N}\theta} \tag{14}$$

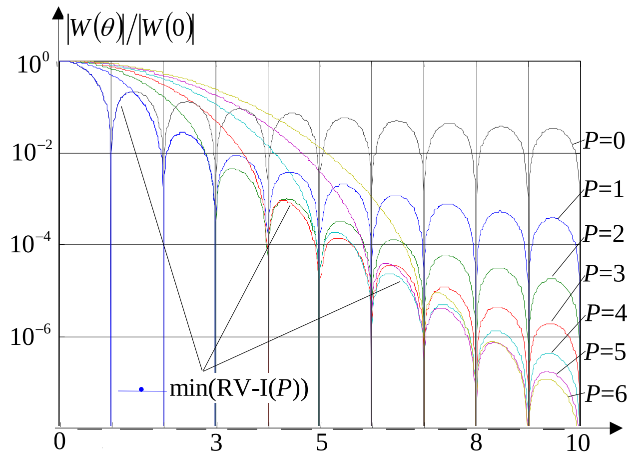
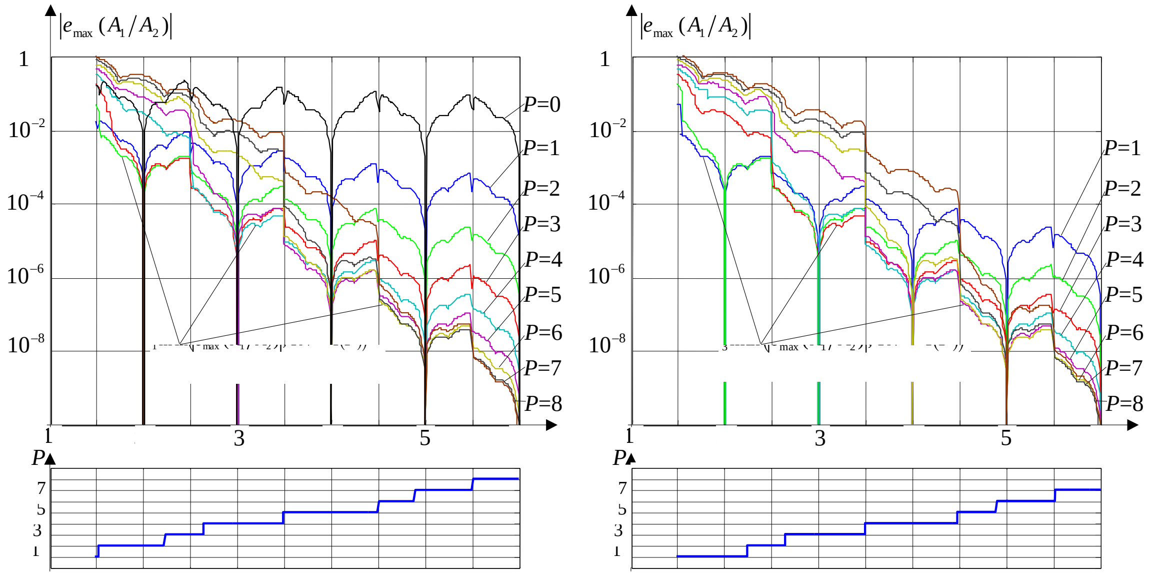


Figure 1. The normalized spectra shapes of the RV-I windows: $P = 0$ – the rectangular window, $P = 1$ – the Hann window, ….

Using suitable order of the RV-I windows in dependence of the relative frequency will give the lowest systematic leakage error in the parameters' estimations. We checked the error of the amplitude quotient estimation $e(A_1/A_2) = (A_1/A_2)/(A_1/A_2)^* - 1$ ($(A_1/A_2)^*$ is the true value of the amplitude quotient) for one sine component in both signals $g_1(t)$ and $g_2(t)$ with a double scan varying both common frequency and the phase of the second signal because the long-range leakages are frequency and phase dependent ($A_1 = 1$; $A_2 = 1$ $N = 1024$; $0.4 \le \theta_1 = \theta_2 \le 6$, $\Delta\theta = 0.01$ and $\varphi_1 = 0$, $-90^\circ \le \varphi_2 \le 90^\circ$, $\Delta\varphi_2 = 5^\circ$). The absolute maximum values of errors (from 37 iterations) at the given relative frequency were compared for the RV-I windows with different orders $P = 0 \div 7$ (Fig. 2: one-point estimation by (5), three-point estimation by (10)).



a) the one-point estimation by (5) b) the three-point estimation by (10)

Figure 2. Maximal relative values of errors of the amplitude quotient estimation using RV-I windows and the curve of the minimal maximal errors dependening on the corresponding window order *P*

Comparing the bias systematic errors (Figs. 2 and 3) shows that RV-I windows perform wider estimation error 'main-lobe' with increasing the order and effectively reduce the long-range errors with increasing frequency θ_m .

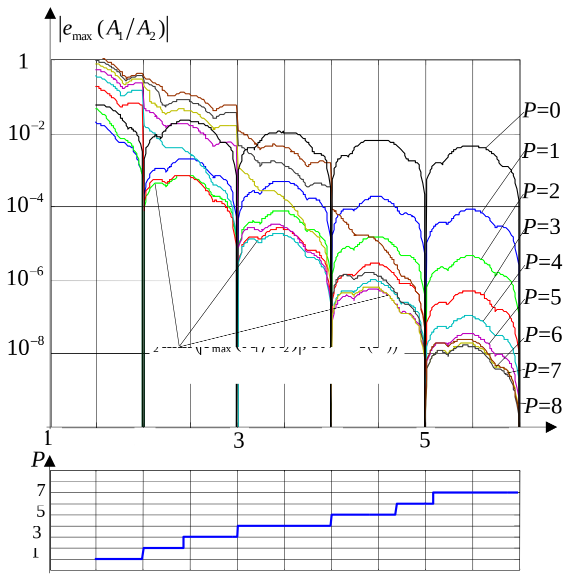


Figure 3. Maximal relative values of errors of the amplitude quotient estimation with the two-point interpolation using RV-I windows by (8) and the curve of the minimal maximal errors dependening on the corresponding window order *P*

Looking for the best performance, the minimal curve of the leakage errors could be found. Fig. 4 shows collected minimal curves of the maximal bias errors of the amplitude quotient estimations with the one-, two-, three-, five- and seven-point interpolations of the DFT using RV-I windows and the corresponding orders $P = 1 \ldots 7$, with which these minimal values have been obtained. The odd numbers of interpolation DFT points (1p, 3p,..) perform the same minimal curves of the maximal errors of the amplitude quotient estimations but the RV-I window order has to be adapted in relation to signal cycles in the measurement interval θ and the number of interpolation points. In the vicinity of $\theta = 2$ the lowest systematic bias errors are achieved by the 1p estimation and order $P = 2$ or combination $3\text{p} \leftrightarrow P = 1$. In the vicinity of $\theta = 3$, this is achieved by combinations:

$1\text{p} \leftrightarrow P=4$ or $3\text{p} \leftrightarrow P=3$ or $5\text{p} \leftrightarrow P=2$, in the vicinity of $\theta=4$, this is achieved by following combinations: $1\text{p} \leftrightarrow P=5$ or $3\text{p} \leftrightarrow P=4$ or $5\text{p} \leftrightarrow P=3$ or $7\text{p} \leftrightarrow P=2$, etc. To achieve the lowest systematic bias errors we can increase the number of the used DFT coefficients in estimations and decrease the order of the RV-I windows or fix the number of the used DFT coefficients (typically three coefficients) and increase the order (Fig. 2). The even numbers of interpolation points perform the same behavior (Figs. 3 and 4).

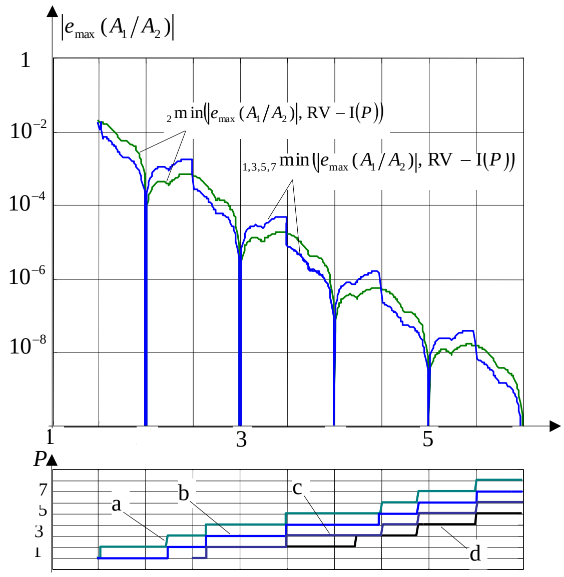
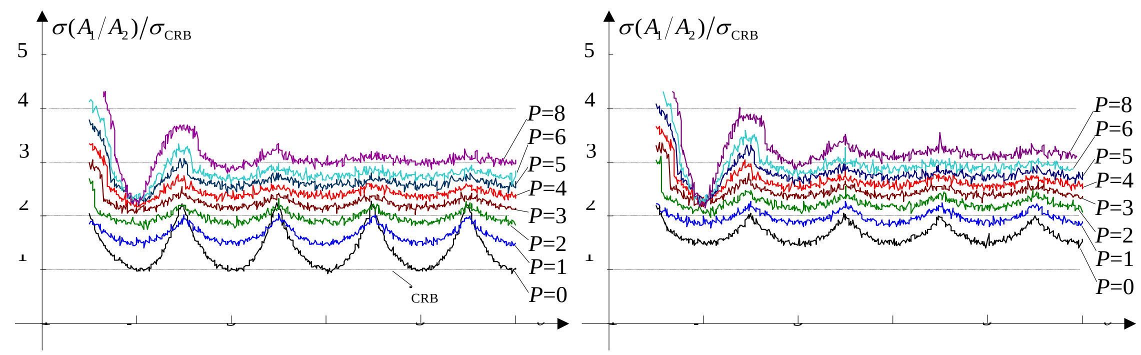


Figure 4. Minimal curves of the maximal errors of the amplitude quotient estimations with interpolations of the DFT using RV-I windows and the coresponding orders $P=1...7$: a - the one-point est., b - the three-point est., c - the five-point est., d - the seven-point estimation

The reduction of the systematic errors increases the contribution of the noise random part of errors [6]. The quantization error is a minimum that has to be taken into consideration in the measurement uncertainty of the final result. In simulations, the white noise with the rectangular distribution, zero mean and the standard deviation $\sigma_t = A_{\text{noise}}/\sqrt{3}$ was added to both signals in the time domain. At every test point with changing frequency and phase like in Fig 2, 50 trials of random added noise were used for the estimation of the quotient standard deviation (Figs. 5 and 6: at every frequency altogether $19 \cdot 50 = 950$ trials). The corresponding signal to noise ratio in the time domain for the particular signal was $SNR_i = A_i^2/(2\sigma_i^2)$. The noise propagations in the algorithms were compared to the Cramér-Rao lower bound [10].

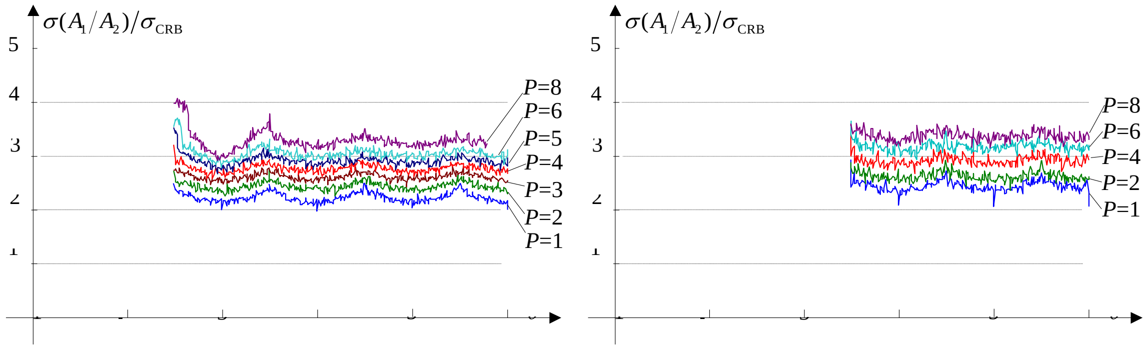
$$\sigma_{\text{CRB}}(A_2/A_1) = A_2/A_1 \sqrt{\frac{SNR_1 + SNR_2}{N\, SNR_1\, SNR_2}} \tag{15}$$



a) the one-point estimation by (5) b) the three-point estimation by (10)

Figure 5. Standar deviations of errors of the amplitude quotient estimation using RV-I windows ; signal 1 and 2: $A_1 = A_2 = 1$, $A_{\text{noise},1} = A_{\text{noise},2} = 0.001$, $SNR_1 = SNR_2 = 1{,}5 \cdot 10^6$ ($61{,}7\,\text{dB}$); $N = 1024$; $\sigma_{\text{CRB}}(A_2/A_1) = 36 \cdot 10^{-6}$

Comparison of the estimation noise errors (Fig. 5) shows that we have the same level of standard deviations with 1p and 3p estimations except that with the last one the order is lower for one: $\sigma_{1p}(P=2)=\sigma_{3p}(P=1)$ or $\sigma_{1p}(P=3)=\sigma_{3p}(P=2)$, etc. Simulations show (Fig. 6) that we can generalize this equal behaviors $\sigma_{1p}(P=4)=\sigma_{3p}(P=3)=\sigma_{5p}(P=2)=\sigma_{7p}(P=1)$ or $\sigma_{1p}(P=5)=\sigma_{3p}(P=4)=\sigma_{5p}(P=3)=\sigma_{7p}(P=2)$, etc. In parallel to systematic error, we can conclude that the lowest systematic bias errors at particular relative frequency have the same noise error level (for example: in the vicinity of $\theta=3$, minimal bias is achieved by combinations: $1\text{p}\leftrightarrow P=4$ or $3\text{p}\leftrightarrow P=3$ or $5\text{p}\leftrightarrow P=2$ (Fig. 4) but noise level is the same $\sigma_{1p}(P=4)=\sigma_{3p}(P=3)=\sigma_{5p}(P=2)$ (Figs. 5 and 6)).



a) the five-point interpolation by (11) b) the seven-point interpolation by (12)

Figure 6. Standar deviations of errors of the amplitude quotient estimation using RV-I windows

I. Conclusions

Single-step algorithms for fast measurement and estimation of the amplitude ratio of the sinusoidal signals with the same frequency from two channels are presented. Parameters are calculated from the DFT coefficients around the component peaks. Paper analyses and compares the systematic bias error and the noise error behaviors changing the order of Rife-Vincent windows class I. The lowest systematic bias errors can be found with increasing the order of RV-I windows when a number of cycles in the measurement time increases. The noise propagations using higher order widows are about 1.5- to 3.5-times larger than the CR lower bound and are displacement dependent.

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