

Hypothesis Test on the ratio between two measurement uncertainties

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Abstract- In a measurement context often it is fundamental to verify if two independent measurement processes give origin to the same uncertainty. The comparison between these two uncertainties is important, for instance when we use two or more different instruments. We apply this method to the uncertainty of a digital oscilloscope measuring rise/fall time of a signal. We apply this test to verify this property if it is requested by other important hypothesis tests like that about the difference between the measurand estimators. This technique is also the base to the ANOVA (ANalysis Of VAriance) procedure. Such statistical tools are widely adopted in the measurement since to draw conclusions from experimental results, see for example [1-3].

I. Testing the ratio between two combined uncertainties

If we consider two measurement processes with normal distribution, we extract two samples of independent measures of size n_1 and n_2 respectively. The variances of these two sample are u_1^2 and u_2^2 where u_1^2 is the greater.

The test to verify null hypothesis $H_0 : u_1^2 = u_2^2$ is based on the ratio between the two combined uncertainties. If these hypothesis are true, it is possible to assume that the statistic:

$$F = \frac{u_1^2}{u_2^2} \quad (1)$$

corresponds to a F Fisher distribution [4,5] with $\nu_1 = n_1 - 1$ and $\nu_2 = n_2 - 1$ degrees of freedom. The critical values, corresponding to the quantiles of the Fisher distribution, depend on the degree of freedom and they can be identified as $F_\alpha(\nu_1, \nu_2)$ and $F_{1-\alpha}(\nu_1, \nu_2)$, respectively, deduced by F distribution. For this distribution we have: $F_\alpha(\nu_1, \nu_2) = 1/F_{1-\alpha}(\nu_2, \nu_1)$. Generally the α -levels commonly used for this test are: $\alpha = 0.01$ and $\alpha = 0.05$ [4]. In Table I the critical values in function of α -level for one-side and two-side test are summarized.

Table I - critical values in function of α -level for one-side and two-side test

Test	Alt. Hyp. H_1	α Level	Critical values	Reject H_0 if:
One side	$u_1^2 > u_2^2$	0.01	$F_\alpha = F_{0.01}$	$F > F_{0.01}$
		0.05	$F_\alpha = F_{0.05}$	$F > F_{0.05}$
One side	$u_1^2 < u_2^2$	0.01	$F_{1-\alpha} = F_{0.99}$	$F < F_{0.99}$
		0.05	$F_{1-\alpha} = F_{0.95}$	$F < F_{0.95}$
Two sides	$u_1^2 \neq u_2^2$	0.01	$F_{1-\alpha/2} = F_{0.995}$	$F < F_{0.995}$
			$F_{\alpha/2} = F_{0.005}$	$F > F_{0.005}$
Two sides	$u_1^2 \neq u_2^2$	0.05	$F_{1-\alpha/2} = F_{0.975}$	$F < F_{0.975}$
			$F_{\alpha/2} = F_{0.025}$	$F > F_{0.025}$

This approach represents an extension of the uncertainty evaluation describes in [6-12].

II. A first case study: DAC rise and fall time, testing the uncertainty ratio

Referring to the measurement setup describe in [13], the method described in the previous Section has been applied to the measurement of rise time and fall time of square waveform produced by different equipment including DAC output sections, i.e. arbitrary waveform generators and data acquisition boards. The measures have been carried out by automating the preset measurement functions of a digital oscilloscope. In particular, the DAC is embedded in an Agilent 33220A arbitrary waveform generator has been used in the measurement set-up. The 10 values of both rise time (S1) and fall time (S2) have been measured by means of the oscilloscope LeCroy SDA 6000 on a square wave having an amplitude of 1V and a frequency of 20 MHz at the sampling frequency of 20GS/s.

We hypothesize that the uncertainties of the oscilloscope rise time / fall time square wave measurement process (S1/S2 signal) are approximatively the same.

The results of the test are reported in Table II. In Figures 1 and 2 the standard deviation for 95% confidence interval , the boxplots and the plot *F*-distribution are shown. All the results are in nanoseconds.

Table II - Test for Equal Variances – S1/S2 Rise Time – Fall Time

95% confidence intervals for standard deviations				
Signal	#Sample	Lower	StDev	Upper
S1	10	0.0640289	0.0978851	0.197107
S2	10	0.0480947	0.0735254	0.148055

F-Test (Normal Distribution)
 Test statistic = 1.77; p-value = 0.407

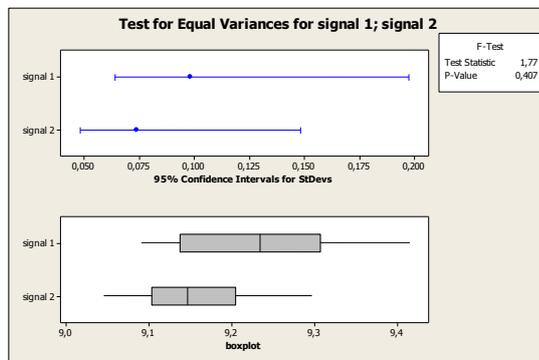


Figure 1 – Standard deviation @95% confidence interval and box-plot for the two signals

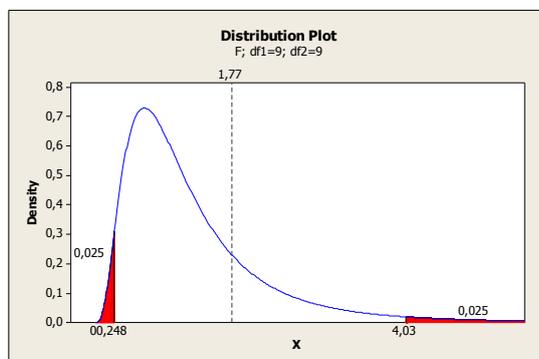


Figure 2 – *F*-distribution plot with the test statistic in the acceptance region

III. Testing the two populations means given large and small samples

Following the previous case, when the variances of the two populations are unknown but tested and assumed equal, $u_1^2 = u_2^2 = u^2$, to test the difference between the two normal populations means, $\mu_1 - \mu_2 = d$, given two large samples, we first find the parameter s^2 as an estimate of u^2 , as follows:

$$s^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2} = \frac{\sum_{i=1}^{n_1} (x_{i1} - \bar{x}_1)^2 + \sum_{i=1}^{n_2} (x_{i2} - \bar{x}_2)^2}{n_1 + n_2 - 2} \quad (2)$$

it is now possible to evaluate the following statistic parameter:

$$z = \frac{\bar{x}_1 - \bar{x}_2 - (\mu_1 - \mu_2)}{s\sqrt{1/n_1 + 1/n_2}} \quad (3)$$

that we can test using the Gaussian distribution.

In the different case when we have to test the difference between the two population means, $\mu_1 - \mu_2$, but given two small samples ($n_1, n_2 \leq 30$), we can use the same s^2 as an estimate of u^2 but now we build the following statistic parameter:

$$t = \frac{\bar{x}_1 - \bar{x}_2 - (\mu_1 - \mu_2)}{s\sqrt{1/n_1 + 1/n_2}} \quad (4)$$

To test again the null hypothesis about $\mu_1 - \mu_2$ but using *t-Student* distribution with $n_1 + n_2 - 2$ degrees of freedom.

So, performing an hypothesis test on the difference between the two populations means with two small samples available and when the two uncertainties are unknown, the test can be carried out in two different phases:

- the first step consist in the verification of the equality of uncertainties by Fisher's test;
- if this equality is verified we perform the *t*-test on the difference between the population mean. In this case we test the null hypothesis $H_0 : \mu_1 - \mu_2 = d$.

In Table III the critical values in function of α -level for one-side and two-side *t*-test are summarized.

Table III - critical values in function of α -level for one-side and two-side *t*-test

Test	Alt. Hyp. H_1	α Level	Critical values	Reject H_0 if:
One side	$\mu_1 - \mu_2 > d$	0.01	$t_\alpha = t_{0.01}$	$T > t_{0.01}$
		0.05	$t_\alpha = t_{0.05}$	$T > t_{0.05}$
One side	$\mu_1 - \mu_2 < d$	0.01	$t_\alpha = -t_{0.01}$	$T < -t_{0.01}$
		0.05	$t_\alpha = -t_{0.05}$	$T < -t_{0.05}$
Two sides	$\mu_1 - \mu_2 \neq d$	0.01	$t_{\alpha/2} = t_{0.005}$	$T > t_{0.005}$
			$t_{\alpha/2} = -t_{0.005}$	$T < -t_{0.005}$
		0.05	$t_{\alpha/2} = t_{0.025}$	$T > t_{0.025}$
			$t_{\alpha/2} = -t_{0.025}$	$T < -t_{0.025}$

IV. A second case study: DAC rise and fall time, testing the expected values

The results of the test are reported in Table IV. In Figures 3 and 4 the individual value and the box-plot for signals S1 and S2 are shown.

Table IV - Test for the difference of the two signal expected values – S1/S2 Rise Time – Fall Time

<i>Mean and standard deviations for both signals</i>				
Signal	#Sample	Mean	StDev	Type A uncertainty
S1	10	9.2331	0.0979	0.031
S2	10	9.1541	0.0735	0.023

*T-Test of difference (Normal Distribution) - Estimate for difference: d= 0,0789
 Test statistic = 2.04; p-value = 0.056 ; DF=18; pooled standard deviation s=0.0866*

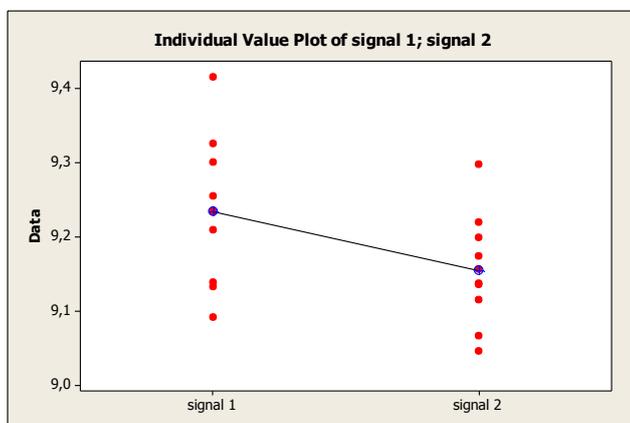


Figure 3 – The line connecting the two means

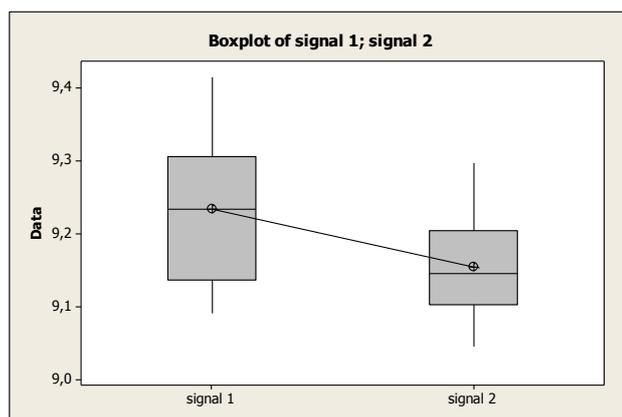


Figure 4 – The line connecting the two means

V. Conclusions

In this paper we test the ratio between two combined uncertainties to apply the *t*-test on the means of the two populations. As discussed in literature [1-5] the inference approach is very useful to obtain more information about the uncertainty assessment.

In the first practical example, a p -value of 0.407 is greater than reasonable choices of α , so we fail to reject the null hypothesis of the variances being equal. That is, these data do not provide enough evidence to claim that the two different populations have unequal uncertainties. Thus, it is reasonable to assume approximately equal the uncertainties when a two-sample t -procedure is used. In the second practical example a p -value of 0.056 is so close to $\alpha=0.05$ that leads to a difficult decision about the rejection of null hypothesis because this decision is strictly dependent on the choice of the α -level so in this case we prefer to reject the null hypothesis. That is, these data do not provide enough evidence to claim that the two different populations have approximately equal expected values.

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VI. References

- [1] M. D'Arco, A. Liccardo, N. Pasquino, "ANOVA-Based Approach for DAC Diagnostics," *IEEE Trans. Instrum. Meas.*, vol. 61, no. 7, pp. 1874-1882, July 2012.
- [2] G. Betta, D. Capriglione, N. Pasquino, "Experimental Investigation on Workers' Exposure to Electromagnetic Fields in Proximity of Magnetic Resonance Imaging Systems," *Measurement*, Vol. 45, Is. 2, February 2012, pp. 199-206.
- [3] M. D'Apuzzo, M. D'Arco, N. Pasquino, "Design of Experiments and Data-Fitting Techniques applied to Calibration of High-Frequency Electromagnetic Field Probes," *Measurement*, vol. 44, no. 6, July 2011, pp. 1153-1165.
- [4] W. Navidi, *Statistics for Engineers and Scientists*, McGraw Hill, NY, USA, 2007.
- [5] M. Catelani, L. Ciani, A. Zanobini, "Uncertainty Interval Evaluation Using The Chi-Square And Fisher Distributions In The Measurement Process", *Metrology and Measurement Systems*, Vol. XVII (2010), No. 2, pp. 195-204.
- [6] E. Balestrieri, M. Catelani, L. Ciani, S. Rapuano, A. Zanobini, "Word Error Rate measurement uncertainty estimation in Digitizing Waveform Recorders", *Measurement* 46 (2013), pp. 572-581, DOI 10.1016/j.measurement.2012.08.016.
- [7] E. Balestrieri, M. Catelani, L. Ciani, S. Rapuano, A. Zanobini, "The Student's t distribution to measure the word error rate in analog-to-digital converters", *Measurement*, Volume 45, Issue 2, February 2012, Pages 148-154, ISSN 0263-2241, DOI 10.1016/j.measurement.2011.03.002.
- [8] E. Balestrieri, M. Catelani, L. Ciani, S. Rapuano, A. Zanobini, "Efficient estimation of the Word Error Rate of Analog-to-Digital Converters", *Proc. Of 2011 IMEKO IWADC & IEEE ADC FORUM*, Orvieto (PG), pp. 221-227.
- [9] E. Balestrieri, M. Catelani, L. Ciani, S. Rapuano, A. Zanobini, "New Statistical Approach to Word Error Rate Measurement in Analog-to-Digital Converters", *Proc. Of 17th Symposium IMEKO TC 4 on Instrumentation for the ICT Era and 15th International Workshop on ADC Modelling and Testing*, Sept. 8-10, 2010, Kosice, Slovakia, pp. 607-613.
- [10] M. Catelani, A. Zanobini, L. Ciani, "Statistical Analysis Of The Word Error Rate Measurement In Analog-To-Digital Converters", *Proc. of XIX IMEKO World Congress Fundamental and Applied Metrology*, September 6-11, 2009, Lisbon, Portugal, pp. 694-697.
- [11] M. Catelani, A. Zanobini, L. Ciani, "Introduction to the t and chi-square distribution for a more accurate evaluation of the measure of the Word Error Rate in Analog-to-Digital Converters", *Metrology and Measurement System*, Issue 4/2008, vol. XV, Number 4 (2008), pp. 483-488, ISSN: 08608229.
- [12] M. Catelani, A. Zanobini, L. Ciani, "Word Error Rate Measurement in Analog-to-Digital Converters: some statistical considerations about IEEE Std 1241-2000, Annex A", *Proc. Of 16th IMEKO TC4 Symposium on Exploring New Frontiers of Instrumentation and Methods for Electrical and Electronic Measurements*, Florence, Italy, Sept. 2008, pp. 1145-1150.
- [13] E. Balestrieri, M. Catelani, L. Ciani, S. Rapuano, A. Zanobini, "Uncertainty evaluation of DAC Time Response -Parameters", *Proc. Of 2011 IMEKO IWADC & IEEE ADC FORUM*, Orvieto (PG), pp. 278-281.