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The quantum limit to incoherent imaging is achieved by linear interferometry

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***Abstract* – We address the problem of estimating the three-dimensional positions of N_S weak incoherent point-like emitters in an arbitrary spatial configuration [1]. We show that a structured measurement strategy in which a passive linear interferometer feeds into an array of photo-detectors is always optimal for this estimation problem, in the sense that it saturates the quantum Cramér-Rao bound. This work provides insights into the phenomenon of super-resolution through incoherent imaging that has attracted much attention recently. Our results may find applications over a broad spectrum of frequencies, from fluorescence microscopy to stellar interferometry.**

I. INTRODUCTION

Quantum imaging [2] exploits quantum features of light to create an image of an object that emits or scatters light. Typically, the goal is to beat the limits of classical imaging. Examples include ghost imaging, quantum lithography, and quantum sensing, which exploit entanglement to enable sensitivity and precision beyond what is achievable classically. A different approach is followed in fluorescence super-resolution microscopy, which exploits carefully engineered emitters and measurements to break the diffraction limit.

Tsang, Nair, and Lu [3] considered the problem of measuring, through imaging, the transverse angular separation between two incoherent sources of identical intensity. They showed that a structured measurement setup is superior to direct imaging. In fact, using direct detection the error in the estimation of the angular separation increases substantially when the separation between the sources falls below the Rayleigh length (this phenomenon was dubbed the "Rayleigh curse"). A number of theoretical and experimental works have been published following the seminal paper of Ref. [3] (see Ref. [4] for a recent review). However, most of these works have focused – with a few exceptions – on the textbook problem of imaging a pair of point-like emitters. There is still no general quantum theory that can be applied to a situation where an arbitrary number of emitters with arbitrary intensities lay within a region of the size of the Rayleigh length. Furthermore, we still lack general insights into why interferometric measurements are optimal for this family of estimation problems. Here we discuss both these issues.

The work of Tsang, Nair, and Lu [3] introduced two important conceptual breakthroughs. The first was to frame imaging as a problem of quantum parameter estimation [5, 6]. This allowed them to use the tools of quantum estimation theory, in particular the quantum Fisher information. The second breakthrough was to consider, instead of direct detection, a more general measurement setup that exploited a multi-mode interferometer. Here we develop the logical consequence of these two conceptual breakthroughs, and show that an interferometric measurement is always the optimal approach to parameter estimation, in the sense that it saturates the quantum Cramér-Rao bound.

II. THE MODEL

Our theory is based on a simple, though general, model for the optical system that is used to collect and measure light. Consider a system of N_S point-like objects that emit or scatter light. We assume that the emitted light is incoherent and quasi-monochromatic. These objects are measured by collecting the light that impinges on a system of N_C collectors (see Fig. 1). For example, these collectors may be physically realised as pin holes on a light-collection plane, microlenses coupled into optical fibres, they could be the inputs of a photonic lantern, or telescope arrays such as in the Very Large Telescope. The collected light is first coherently processed in a multi-port interferometer R, and then measured using N_C photodetectors. We further assume that the position of these emitters is known with great precision.

Let us label the emitters with the parameter $s = 1, 2, \ldots, N_S$. The s-th emitter has spatial coordinates

$$r_s = (x_s, y_s, z_0 + z_s) \,, \tag{1}$$

where the first two are the transverse coordinates and the third component lies along the optical axis, and z_0 is a reference distance between the objects and the collection plane. For the sake of simplicity, we assume that the collectors lie in a transverse plane, and the j-th collector has spatial coordinates

$$w_j = (u_j, v_j, 0) \,. \tag{2}$$

We work in the limit of weak sources and assume that at most one photon is collected in a single detection window. Under these assumptions, we describe the state of light using a single-photon wave function.

The state of a photon emitted by the s-th source, that impinges on the N_C collectors, is described by the single-photon wave function:

$$|\psi(r_s)\rangle = \sum_{j=1}^{N_C} \gamma(w_j, r_s)|j\rangle\,. \tag{3}$$

Here $|j\rangle$ denotes the state of a photon arriving at collector j, and $\gamma(w_j, r_s)$ is the corresponding complex amplitude. This model requires that each collector couples with one mode only. In general, the phase of $\gamma(w_j, r_s)$ is expressed by the optical path length from the source to the collector:

$$\arg\gamma = ik\sqrt{(x_s-u_j)^2+(y_s-v_j)^2+(z_0-z_s)^2}\,, \tag{4}$$

where k is the wave number. The modulus of γ is inversely proportional to the distance between the source and the collector. The normalization condition is

$$\sum_j |\gamma(w_j, r_s)|^2 = 1\,. \tag{5}$$

Since only one photon is detected, but we do not know which source emitted it, the average state is represented by the following density operator:

$$\rho(r) = \sum_{s=1}^{N_S} p(s)|\psi(r_s)\rangle\langle\psi(r_s)|\,, \tag{6}$$

where

$$r \equiv (x_1, y_1, z_1, \ldots, x_{N_S}, y_{N_S}, z_{N_S}) \tag{7}$$

indicates the $3N_S$ collective coordinates of the N_S emitters, and $p(s)$ is the probability that the photon is emitted by source s.

Consider a unit vector with $3N_S$ components

$$a = (a_1, a_2, \ldots, a_{3N_S})\,. \tag{8}$$

A generalized coordinate ϑ is defined as

$$\vartheta := a \cdot r\,, \tag{9}$$

i.e., the scalar product of a unit vector with the collective coordinates of the object. For a small variation $\delta\vartheta$, the collective coordinate changes from r to

$$r' = r + a\delta\vartheta\,, \tag{10}$$

with

$$\delta\vartheta = a\cdot(r'-r)\,. \tag{11}$$

We are interested in estimating the value of a generalized coordinate of the system of N_S emitters by collecting and measuring the light impinging on the N_C collectors. To quantify the ultimate precision limit of such an estimation, we compute the quantum Fisher information $I_Q(\vartheta)$ within our model. Second, we compute the classical Fisher information $I_C(\vartheta)$ for a specific measurement strategy where R is a linear interferometer followed by photo-detectors. Finally, we show that there exists a choice of the interferometer that saturates the quantum Fisher information. This implies the optimality of a measurement strategy based on a multi-port interferometer followed by photo-detection.

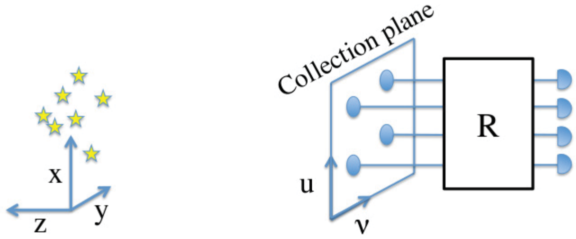

Fig. 1. A collection of N_S incoherent point-like emitters (left) and the apparatus used to measure them (right). The light emitted or scattered by the objects is collected at N_C specific locations in the collection plane. The collected light is first coherently processed in a N_S-port interferometer, and then measured by photo-detection.

III. METHODS

Let us recall that the quantum Fisher information is used to quantify the minimal statistical error in the estimation of the parameter ϑ. For comprehensive reviews please see Refs. [5, 6]. In our setting, we collect the light coming from the object using the N_C collectors. Each collector is described as a single optical mode. The state of the light impinging on the collectors is represented by the density operator $\rho(r)$ in Eq. (6), where r is related to ϑ through Eq. (9). If we repeat the experiment n times with identical initial conditions, then the overall state is represented as the n-fold tensor power

$$\rho(r)^{\otimes n} = \underbrace{\rho(r)\otimes\rho(r)\otimes\ldots\rho(r)}_{n}\,. \tag{12}$$

A collective measurement is a measurement that is applied to this n-fold quantum state. Such a measurement will output an estimate $\hat{\vartheta}$ of the unknown parameter ϑ. The estimate $\hat{\vartheta}$ is a random variable, and its variance $\Delta\vartheta^2$ quantifies the statistical error associated with the measurement. The estimator is said to be unbiased if

$$\langle\hat{\vartheta}\rangle = \vartheta\,, \tag{13}$$

where $\langle\hat{\vartheta}\rangle$ is the expectation value of the random variable $\hat{\vartheta}$. According to the Cramér-Rao bound, $\Delta\vartheta^2$ cannot be arbitrary small, but it is bounded from below as

$$\Delta\vartheta^2 \geq \frac{1}{n}\frac{1}{I_Q(\vartheta)}\,, \tag{14}$$

where $I_Q(\vartheta)$ is the quantum Fisher information. This bound holds for any measurement strategy, as long as the estimator is unbiased.

The quantum Fisher information for the estimation of the parameter ϑ from a measurement of the quantum state ρ is given by the following expression:

$$I_Q(\vartheta) = \mathrm{Tr}\left(\mathcal{L}^2\rho(r)\right)\,, \tag{15}$$

with

$$\mathcal{L} = \sum_{j,k:\lambda_j+\lambda_k>0} \frac{2}{\lambda_j+\lambda_k}\,\langle\lambda_j|\frac{\partial\rho(r)}{\partial\vartheta}|\lambda_k\rangle|\lambda_j\rangle\langle\lambda_k|\,, \tag{16}$$

where the sum is over the eigenvalues of ρ, denoted as λ_j, and $|\lambda_j\rangle$ are the corresponding eigenvectors. Note that the quantum Fisher information is additive, therefore, given the tensor power state $\rho(r)^{\otimes n}$, it can be computed for any n following a diagonalisation of the density operator $\rho(r)$.

The power of the Cramér-Rao bound is that it is independent on the particular measurement strategy employed. Furthermore, under fairly general conditions, a measurement exists that saturates the bound in the limit of large n. However, finding an explicit and feasible measurement that saturates the Cramér-Rao bound is a highly non-trivial task. A generalised measurement (POVM: Positive Operator-Values Measurement) is described by a set of non-negative operators Λ_y, for $y = 1,\dots,m$, such that $\sum_y \Lambda_y = I$, with I the identity operator. The probability of obtaining the outcome y is given by the Born rule:

$$p_\vartheta(y) = \mathrm{Tr}\left(\Lambda_y\rho(r)\right)\,, \tag{17}$$

which depends on ϑ through $\rho(r)$. The classical Fisher information $I_C(\vartheta)$ bounds the statistical error in the estimation of ϑ, using the data obtained from the given measurement, through the (classical) Cramér-Rao bound:

$$\Delta\vartheta^2 \geq \frac{1}{n}\frac{1}{I_C(\vartheta)}\,, \tag{18}$$

with

$$I_C(\vartheta) = \sum_y p_\vartheta(y)\left(\frac{\partial\log p_\vartheta(y)}{\partial\vartheta}\right)^2\,. \tag{19}$$

As the quantum Cramér-Rao bound is measurement-independent, for any given POVM we have

$$I_C(\vartheta) \leq I_Q(\vartheta)\,. \tag{20}$$

Here we will use alternative expressions for the quantum and classical Fisher information. For the quantum Fisher information we will use the following characterisation:

$$I_Q(\vartheta) = \lim_{\delta\vartheta\to 0}\frac{8\left(1-F(\rho(r),\rho(r'))\right)}{\delta\vartheta^2}\,, \tag{21}$$

where

$$F(\rho(r),\rho(r')) = \mathrm{Tr}\left|\sqrt{\rho(r)}\sqrt{\rho(r')}\right| \tag{22}$$

is the quantum fidelity between the states $\rho(r)$ and $\rho(r')$, recall that $r' = r + a\delta\vartheta$. This characterisation holds as long as the states $\rho(r)$ and $\rho(r')$ have the same rank [7]. Similarly, for the classical Fisher information we will use the expression

$$I_C(\vartheta) = \lim_{\delta\vartheta\to 0}\frac{8\left(1-F_c(p_\vartheta,p_{\vartheta+\delta\vartheta})\right)}{\delta\vartheta^2}\,, \tag{23}$$

where

$$F_c(p_\vartheta,p_{\vartheta+\delta\vartheta}) = \sum_y \sqrt{p_\vartheta(y)p_{\vartheta+\delta\vartheta}(y)} \tag{24}$$

is the classical fidelity between the probability distributions p_ϑ and $p_{\vartheta+\delta\vartheta}$.

IV. RESULTS

Here we present a number of results:

1. We obtain an expression for the quantum Fisher information for the estimation of an arbitrary collective coordinate ϑ;
2. We obtain an expression for the classical Fisher information that is obtained for a specific measurement strategy, comprising a N_C-mode interferometer followed by photo-detection;
3. We show that there exists a choice of interferometer that saturates the quantum Fisher information.

In conclusions, this shows that linear optics followed by photo-detection is an optimal measurement strategy to estimate a collective coordinate of N_S point-like emitters.

The quantum Fisher information. Following Eq. (21), the quantum Fisher information can be obtained from the calculation of the fidelity. For a pair of pure states ψ, ψ', the fidelity reads

$$F(\psi,\psi') = |\langle\psi|\psi'\rangle|\,. \tag{25}$$

For mixed state, it can be written in terms of the fidelity of their purifications. First extend the mixed states $\rho(r)$ and $\rho(r')$, with the help of an auxiliary quantum system A, into the pure states $\Psi(r)$, $\Psi(r)$. These states are purifications if they satisfy

$$\mathrm{Tr}_A(\Psi(r)) = \rho(r)\,, \mathrm{Tr}_A(\Psi(r')) = \rho(r')\,. \tag{26}$$

Then the fidelity can be written as

$$F(\rho(r), \rho(r')) = \max_V |\langle \Psi(r)| I \otimes V |\Psi(r')\rangle| \,, \tag{27}$$

where the maximisation is over the unitary V acting on the ancillary system A.

To compute the quantum Fisher information for the estimation of ϑ we write the following purification of the mixed state $\rho(r)$ in Eq. (6):

$$|\Psi(r)\rangle = \sum_{j=1}^{N_C} \sum_{s=1}^{N_S} c(\omega_j, r_s) |j\rangle |s\rangle \,, \tag{28}$$

where

$$c(\omega_j, r_s) = \sqrt{p(s)}\gamma(\omega_j, r_s) \,, \tag{29}$$

and the purifying ancillary system is represented as a N_S-dimensional Hilbert space. The unitary transformation V has matrix elements $V_{st} = \langle s|V|t\rangle$ in the given basis of the auxiliary system. Therefore the following holds:

$$F(\rho(r), \rho(r')) = \max_V |\langle \Psi(r)| I \otimes V |\Psi(r')\rangle| \tag{30}$$

$$= \max_V \left| \sum_{st} \sum_j c^*(\omega_j, r_s) c(\omega_j, r'_t) \right| \tag{31}$$

$$= \max_V \left| \mathrm{Tr}(V^{\mathsf{T}} M) \right| \,, \tag{32}$$

where we have introduced the matrix M with components

$$M_{st} = \sum_j c^*(\omega_j, r_s) c(\omega_j, r'_t) \,. \tag{33}$$

Finally, we recall that

$$\max_V \left| \mathrm{Tr}(V^{\mathsf{T}} M) \right| = \|M\|_1 \,, \tag{34}$$

where

$$\|M\|_1 := \mathrm{Tr}|M| = \mathrm{Tr}\sqrt{M^\dagger M} \tag{35}$$

is the trace-norm of the matrix M. In conclusions, for any configuration of N_S point-like emitters and N_C single-mode collectors, the quantum Fisher information is obtained from the trace norm of the matrix M as

$$I_Q(\vartheta) = \lim_{\delta\vartheta \to 0} \frac{8\,(1 - \|M\|_1)}{\delta\vartheta^2} \,. \tag{36}$$

The classical Fisher information. Here we focus on the particular family of measurements obtained by concatenating a multi-mode interferometer with photo-detection. The light emitted from the s-th source, expressed by the state in Eq. (28), is passed through a N_C-mode interferometer R. This transforms the state into

$$R|\psi(r_s)\rangle = \sum_{i,j=1}^{N_C} R_{ij} \gamma(w_j, r_s) |i\rangle \,, \tag{37}$$

where $R_{ij} = \langle i|R|j\rangle$ is the (unitary) $N_C \times N_C$ matrix that characterises the interferometer. Given a distribution r of the sources, the probability that a photo-detection event is recorded on mode i is given by

$$p_r(i|s) = \left| \sum_{j=1}^{N_C} R_{ij} \gamma(w_j, r_s) \right|^2 \,. \tag{38}$$

Summing over all N_S incoherent sources we obtain the overall probability of photo-detection on mode i:

$$p_r(i) = \sum_{s=1}^{N_S} \left| \sum_{j=1}^{N_C} R_{ij} c(w_j, r_s) \right|^2 \tag{39}$$

(recall that $c(w_j, r_s) = \sqrt{p(s)}\gamma(w_j, r_s)$, where $p(s)$ is the probability that the photon is emitted by the s-th source).

We are interested in the (classical) fidelity between the probabilities p_r and $p_{r'}$, for $r' = r + a\delta\vartheta$. This reads

$$F_c(p_r, p_{r'}) = \sum_i \sqrt{ \sum_{s=1}^{N_S} \left| \sum_{j=1}^{N_C} R_{ij} c(w_j, r_s) \right|^2 } \tag{40}$$

$$\times \sqrt{ \sum_{s'=1}^{N_S} \left| \sum_{j'=1}^{N_C} R_{ij'} c(w_{j'}, r'_{s'}) \right|^2 } \,. \tag{41}$$

Cauchy–Schwarz inequality. Equation (20) implies

$$F(\rho(r), \rho(r')) \leq F_c(p_r, p_{r'}) \,. \tag{42}$$

Now we show that this latter inequality is an instance of the Cauchy–Schwarz inequality. To show this, note that the matrix M can be written as

$$M = C(r)^\dagger C(r') \,, \tag{43}$$

where $C(r)$ is the matrix with elements

$$C_{ij}(r) = c(w_i, r_j) \,, \tag{44}$$

and similarly $C_{ij}(r') = c(w_i, r'_j)$. The trace norm $\|M\|_1$ can be obtained from the singular value decomposition of M, i.e. by finding the unitary transformations V and W that make M diagonal:

$$V^\dagger M W = D \,, \tag{45}$$

where D is a diagonal, non-negative matrix, yielding $\|M\|_1 = \mathrm{Tr}D$. It follows that, for any given unitary matrix R,

$$\|M\|_1 = \mathrm{Tr}D = \mathrm{Tr}(V^\dagger M W) \tag{46}$$

$$= \mathrm{Tr}(V^\dagger C(r)^\dagger C(r') W) \tag{47}$$

$$= \mathrm{Tr}(V^\dagger C(r)^\dagger R R^\dagger C(r') W) \tag{48}$$

$$= \sum_{s,s',j,j',t,v} V^*_{st}\, c(w_j, r_s)^* R^*_{vj} R_{vj'} c(w_{j'}, r'_{s'}) W_{s't} \,, \tag{49}$$

where in the third line we have used $RR^\dagger = I$.

For every value of the index v in Eq. (49), consider the vectors $X(v)$ and $Y(v)$, whose components are

$$X_t(v) = \sum_{s,j} V^*_{st}\, c(w_j, r_s)^* R^*_{vj}\,, \tag{50}$$

$$Y_t(v) = \sum_{s',j'} W_{s't} c(w_{j'}, r'_{s'}) R_{vj'}\,. \tag{51}$$

It follows that

$$\|M\|_1 = \sum_v X(v) \cdot Y(v)\,. \tag{52}$$

By applying the Cauchy–Schwarz inequality (for each value of v, and then summing over v) we obtain

$$\|M\|_1 = \sum_v X(v) \cdot Y(v) \leq \sum_v |X(v)||Y(v)| \tag{53}$$

$$= \sum_v \sqrt{\sum_t \left| \sum_{j,s} R_{vj} c(w_j, r_s) V_{st} \right|^2} \times \sqrt{\sum_{t'} \left| \sum_{j',s'} R_{vj'} c(w_{j'}, r'_{s'}) W_{s't'} \right|^2}\,. \tag{54}$$

Note that the quantity in (54) is invariant under the unitary transformations V and W. Therefore, putting $V = W = I$ we finally obtain:

$$\|M\|_1 \leq \sum_v \sqrt{\sum_t \left| \sum_j R_{vj} c(w_j, r_t) \right|^2} \times \sqrt{\sum_{t'} \left| \sum_{j'} R_{vj'} c(w_{j'}, r'_{t'}) \right|^2} = F_c(p_r, p_{r'})\,. \tag{55}$$

In conclusions, we have shown that inequality (20) follows from the Cauchy–Schwarz inequality. Therefore, the optimal interferometric measurement corresponds to a particular choice of the unitary matrix R that saturates this particular instance of the Cauchy–Schwarz inequality. In Ref. [1] we present a constructive proof of the existence of this optimal measurement.

V. CONCLUSIONS

Our analysis shows that linear interferometry and photodetection are always optimal for estimating the (generalised) coordinates of a system of point-like emitters. This result holds in the regime of weak signals. An explicit construction for the optimal interferometer is presented in [1], which can be implemented using standard methods [8]. Our results explain why coherent detection overcomes the Rayleigh curse by recasting imaging as interferometry at the outset.

Further results that specialise these findings to the paraxial regime are presented in Ref. [1]. For example, for the case of two incoherent sources and within the paraxial approximation, the optimal interferometer can have relatively low complexity, for example, a single beam-splitter, or a quantum Fourier transform for the case of four collectors.

A number of important questions remain open. For example: under what general conditions are the optimal interferometer independent of the parameter to be estimated? When is it possible to use the same interferometer for the optimal estimation of multiple parameters?

VI. *

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