2019 IMEKO TC-4 International Conference on
Metrology for Archaeology and Cultural Heritage
Florence, Italy, December 4-6, 2019

A short tale of the short story of the sliding rule

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***Abstract* – The ideation and the design of scientific instruments for measuring and computing have been (and actually they are) often promoted by motivations of strong social and economic interest. In turn, the improvements of instruments influenced the development of new theories. An emblematic example is the story of the sliding rule. In this note, we illustrate the historical origin of the sliding rule, the underlying mathematical principle, and how it can be used to perform arithmetic computations.**

I. INTRODUCTION

The sliding rule is a computing instrument. Schematically, it can be thought of as a pair of straight lines endowed with a logarithmic scale. One line is fixed, while the other is free to slide over the first one. It enables us to obtain approximate results of multiplications of arbitrary real numbers. Since the second half of the 19th century and for about 120 years, it had a wide diffusion among the scientists and technicians community. At that time, mechanical calculators did exist, but they were very expensive, heavy, delicate and difficult to use. On the contrary, a sliding rule is cheap, easily portable and easy to handle. The precision of the approximation achievable by a sliding rule depends on the ability and experience of the user. After all, for large numbers even with a calculating machine the approximation is inevitable, because of the limited number of digits.

The invention of the sliding rule is attributed to Edmund Gunter (1620), with further improvements by William Oughtred (1622), and it dates back a few years later the publication of the book [4], by John Napier. The basic idea is that by means of logarithms, multiplications are transformed into additions, and additions can be realized "geometrically" by juxtaposition of segments of appropriate length. The sliding rule remained almost unknown for more than two centuries, but after the publication of the handbook [5] by Quintino Sella, it became rapidly an indispensable tool for engineers, physicists and mathematicians, in a period when the demand for technological progress was more and more increasing.

To the sudden success of the sliding rule corresponded an equally sudden decline when, around 1970, cheap pocket electronic calculators appeared in the commerce.

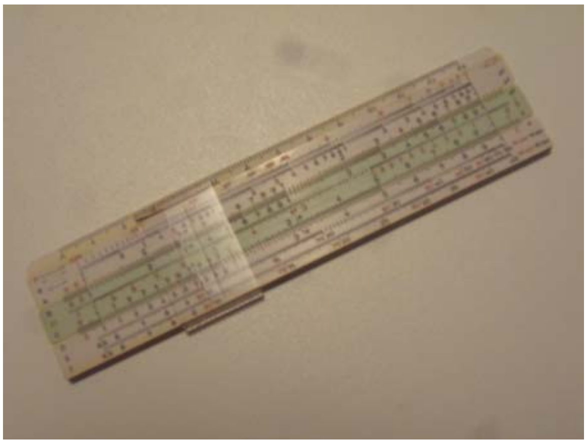


Fig. 1. Pocket sliding rule for scientific use

II. COMPUTING BEFORE LOGARITHMS

For human beings, doing arithmetic operations on papers is not an easy task. The risk of accidental errors, due to tiredness or lack of attention, is very high, especially when many, tedious operations must be performed repeatedly, quickly and with large numbers. Moreover, accidental errors are difficult to detect, and their consequences on the final result unpredictable.

For many centuries, the unique available computing instrument was the abacus. But the abacus is a valid help for additions, not for multiplications (actually, since the multiplication algorithm is a repetition of additions, one can also use the abacus for doing multiplications, but this way is time-consuming and cumbersome; moreover, one needs to use anyway paper and pencil in order to recorder the partial results).

It should be noticed that for human beings, multiplications and divisions are more difficult than additions (according to the common experience and feeling of every student, and not only). Indeed, in the Middle Ages most people with a basic education were able to do additions, but for multiplications it was necessary to address to specialists, called *abacists*, and to pay for it.

At the beginning of 17th century, in some scientific fields the need of complex computations began to emerge. One of this fields is astronomy. There was a strong motivation for this. Indeed, an accurate information about position and motion of stars and planets is necessary for safe ocean navigation, and ocean navigation was in turn essential to guarantee the development of commerce (as witnessed by the famous longitude problem, see [6]). At that time, the most advanced center for astronomic observations was a Danish island, where Ticho Brahe had his laboratory.

Astronomic measures are essentially measures of angles. To elaborate the observed data, Ticho Brahe made an extensive use of the trigonometric prosthaphaeresis formulæ and their inverse

$$\cos\alpha \cdot \cos\beta = \frac{1}{2}\Big(\cos(\alpha+\beta) + \cos(\alpha-\beta)\Big)$$

in order to transform a product into a sum.

III. THE LOGARITHMS

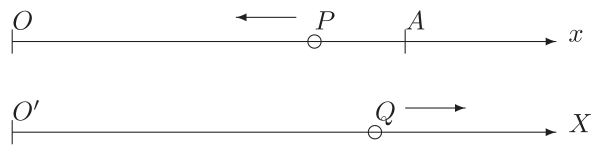
As it frequently happens in the history of sciences, the invention of logarithms is controversial. As a matter of fact, the first published work about logarithms is due to John Napier [4]: in this book the term *"logarithm"* appears for the first time.

Napier spent a large part of his life in Edinburgh. He was a rich Scottish man, and an interested scholar in philosophy, theology and sciences. He was aware of the theory of geometric progression (another topic involving relationship between additions and multiplications) and when he knew, by chance, about Ticho Brahe's work (see the very interested narration reported in [2], Ch. 16, Sect. 9), he realized that a more general approach could be developed.

At a first sight, Napier's approach to logarithms may appear rather surprising. He writes:

The logarithme, therefore, of any sine is a number very neerely expressing the line which increased equally in the meene time whiles the line of the whole sine decreased proportionally into that sine, both motions being equal timed and the beginning equally shift.

Apart from the reference to the "sine" (to be put in relation with the astronomic applications which were, as already noticed, of primary interest at that time), the reasoning can be interpreted in the following way (according to [3], p. 149). Let us consider two half-lines whose origins are O and O', respectively.



On the first line, a body P is situated, at the beginning, at a point A and moves toward the origin. Its velocity decreases, proportionally to its distance from the origin. Simultaneously, a body Q, initially placed at O', moves along the second half line with constant velocity v.

Clearly, the body Q works like a kind of clock. Measuring the positions of P and Q at regular time intervals, Napier notices that:

(a) the abscissa X of Q changes according to an arithmetic progression;

(b) the abscissa x of P changes according to a geometric progression.

Napier concludes that there must be a constant $\beta > 0$ such that

$$x = a\beta^X \tag{1}$$

where a is the abscissa of the initial point A. Napier writes β as

$$\beta = 1 - \frac{1}{v}\,. \tag{2}$$

Then, in order to model appropriately the shape of the curve (1), and to make easier and efficient practical computations, he takes $a = v = 10^7$, so that $\beta = 1 - \frac{1}{10^7} = 0.9999999$. Finally, he claims that

X is the logarithm of x

All that seems to be quite mysterious, but it can be easily explained by revisiting the problem in modern terms. The respective motions of P and Q can be described by the differential system

$$\begin{cases} \dot{x} = -x \\ \dot{X} = 10^7 \end{cases} \tag{3}$$

(where the dot denotes the derivative with respect to time). Solving this system with the initial conditions $x(0) = 10^7$, $X(0) = 0$ we get

$$x = 10^7 e^{-t}\,, \quad X = 10^7 t$$

(here, $e = 2,7182818284...$ is the irrational number today called the Napier number). In the relations above the time t can be eliminated. It is straightforward to recover

$$x = 10^7 e^{-\frac{X}{10^7}} \tag{4}$$

Now, it is well known that

$$\lim_{v\to\infty}\left(1 - \frac{1}{v}\right)^{-v} = e$$

which intuitively means that for large v, the expression $\left(1-\frac{1}{v}\right)^{-v}$ approaches the limit value e. Now, we readily recognize in (1) a very good approximation of (4). Actually, with Napier's choice $v = 10^7$

$$\left(1-\frac{1}{10^7}\right)^{-10^7} = 2.718281962$$

which provides a value of e correct up to 6 decimal figures!

It is worthwhile to stress that at Napier's time, the differential and integral calculus was not yet invented.

In modern notation (due to L. Euler), the logarithm of a positive real number x in base b is defined as

$$y = \log_b x \quad \text{if and only if} \quad b^y = x$$

(here b is any fixed positive number, $b \neq 1$) and what Napier calls the *Mirifici logarithmorum canonis* writes

$$\log_b x_1 x_2 = \log_b x_1 + \log_b x_2 \qquad (5)$$

for each pairs of positive real numbers x_1, x_2. It is curious to point out that the base b of Naperian logarithms is an approximation of $1/e$. Today, the more frequently values chosen for the base are:

• base e (natural logarithms, for scientific purposes); the official ISO notation is ln x

• base 10 (common logarithms, for technical and commercial purposes); the official ISO notation is lg x

Naperian logarithms have some "drawbacks", unexpected in our modern perspective: for instance, the logarithm of one is not zero, and the logarithm of the base is not one.

IV. TABLES OF LOGARITHMS

Of course, in order to apply (5) in general, one needs to know the value of the logarithm of an arbitrary number. To overcome this difficulty, Napier included in [4] a table of logarithms of certain trigonometric quantities referred to the aforementioned base $b \sim 1/e$.

A few years later, H. Briggs replaced the value of b by the more familiar base 10, and published a new table of logarithms of integers from 1 to 1000, with 14 decimal digits. Briggs' method to obtain logarithms consisted in the iterative computation of roots

$$x = 10^{\frac{p}{2^n}} = \sqrt{...\sqrt{10^p}} \Longleftrightarrow \frac{p}{2^n} = \log_{10} x \ .$$

He made use of linear interpolation, as well.

A new list of logarithms with 5 decimal digit was published by Briggs in 1624. This list covered two separate intervals of integers, from 1 to 20000 and from 90000 to 100000. However, for a long time, the most important reference about logarithms was the book by Adriaan Vlacq [8]. Vlacq's logarithms have 10 decimal digits and cover the whole interval from 1 to 100000.

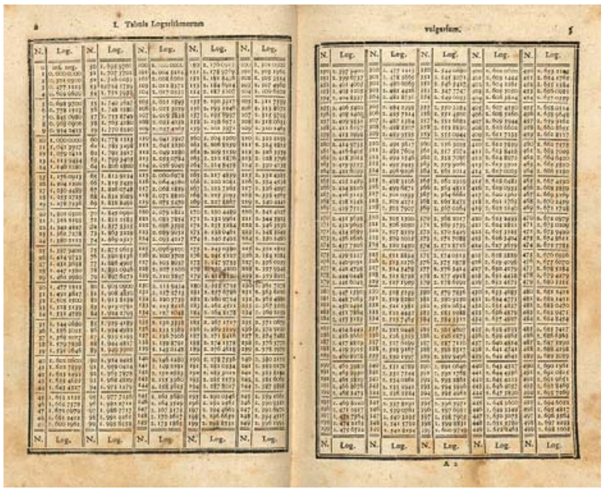


Fig. 2. Old logarithmic tables

Much more efficient methods to compute logarithms came with the introduction of numerical series. The basic facts, in this direction, are:

(1) if the logarithms of all the prime numbers are known, it is possible to compute the logarithm of each other integer number by means if its factorization by applying repeatedly (5);

(2) if the logarithms of the prime numbers up to M are known, it is possible to compute the logarithms of other prime numbers $N > M$ by the series (J. Gregory, 1668)

$$\begin{aligned}\log N &= \frac{\log(N-1)+\log(N+1)}{2} \\ &+ \left[\frac{1}{2N^2-1}+\frac{1}{3(2N^2-1)^3}\right. \\ &+ \left.\frac{1}{5(2N^2-1)^5}+\dots\right] .\end{aligned}$$

Indeed, if N is prime, $N-1$ and $N+1$ are not, and hence they can be factorized by prime numbers less than M. Their logarithms can be computed through this factorization.

Since their introduction, logarithms played a fundamental role in the history of computation. It is remarkable that the continuous revision of existing tables and the construction of new tables more and more feasible, precise and complete had the financial support of the government of

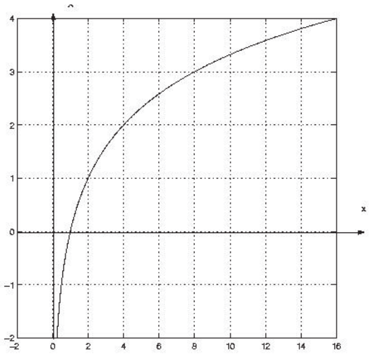


Fig. 3. The logarithmic curve (base 2)

most important States. The agency NIST (National Institute for Standardization and Technology) of the USA government was active in this field at least until 1940.

The sliding rule provides a way to apply logarithms in order to transform products into sums without need of consulting tables.

V. THE LOGARITHMIC SCALE

A logarithmic scale is a special system of abscissæ on the positive real line induced by the logarithmic function $y = f(x) = \log_b x$. Its construction can be described in the following way.

Assume that a usual (Cartesian) system of abscissæ has been fixed on a geometric straight line, so that every point is identified by a real number y. Then, for each point, replace the label y by the label $x = b^y$. To understand the peculiarity of a logarithmic scale, the following remark is in order. Consider four points P_1, P_2, P_3, P_4 on the line, taken in this order, and let respectively y_1, y_2, y_3, y_4 their usual abscissæ. Clearly, the segments P_1, P_2 and P_3P_4 are equal if and only if $y_2 - y_1 = y_4 - y_3$. But, if we refer to the logarithmic abscissæ x_1, x_2, x_3, x_4, then we have that P_1, P_2 and P_3P_4 are equal if and only if

$$\frac{x_2}{x_1} = \frac{x_4}{x_3} .$$

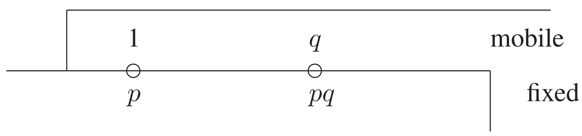
Logarithmic scales are often unconsciously used in some subjects of general interests: for instance, the Richter scale of earthquake's magnitude is a logarithm scale, and the equally tempered music scale is a logarithm scale.

VI. MULTIPLICATION ON THE SLIDING RULE

As already mentioned, a sliding rule in its simpler version is a pair of strips endowed with identical logarithmic scales. One strip is called *fixed*, the other *mobile*. Initially, the fixed and mobile strips are aligned in such a way that the points marked by 1 coincides. Assume we want to find the result of the multiplication of factors a and b. First of all, we write a and b in scientific (normalized floating point) notation

$$a = \pm p \cdot 10^N \quad \text{and} \quad b = \pm q \cdot 10^M$$

where the significants $p, q \in [1, 10)$. The exponent of the result is $N + M$, and the sign is determined according to the well known rule. Thus, our task is reduced to compute $p \cdot q$. To this end, let us slide the mobile strip over the fixed one until the point marked 1 (read on the mobile) is aligned with the point marked p (read on the fixed). The result can be read on the fixed, at the position corresponding to q (read on the mobile).



We already noticed that the sliding rule provides only approximate results of multiplications, and other operations. The degree of precision depends on the dimensions of the sliding rule and the accuracy of the scale. Of course, it depends also on the ability and experience of the operator (and on how good is his/her eyesight). Generally, with a pocket sliding rule, it is hard to achieve more than 2 or 3 exact decimal figures for the significant. In this sense, we can see an analogy with the abacus: the time employed to do a sum on the abacus and the confidence in the result may sensibly vary for expert or not expert users. Never-

Fig. 4. Sliding rule for teaching use

theless, the sliding rule has been a fundamental tool which contributed to the progress of sciences and technologies for more than one century. In that period, learning how to compute by the sliding rule was an important component of the training in the engineering faculties. Figure 4 shows a sliding rule for teaching use, more than 2 meters long, to be fixed to the wall, which has been used for classroom lesson at the Politecnico di Torino for many years, and now conserved at the Department of Mathematical Sciences.

VII. OTHER FUNCTIONS

Multiplication remains the main goal of the sliding rule. However, starting from the half of 19th century new scales were progressively added for other types of operations. The most popular methods to equip a sliding rule with appropriate scales were the *Darmstadt method*, very diffused in the academic community, and the *Rietz method* for more technical purposes.

In both methods, the fundamental scales to be used for multiplications are often labeled by the letter D (on the fixed) and C (on the mobile). The square and the cube of a number marked on the scale D can be directly read on the scale A (or B) and K, respectively.

Sliding rules endowed with the Darmstadt method present, on the back of the mobile, scales labeled LL1, LL2, LL3 where the values of exponentials (or, reversing the procedure, logarithms) can be read. Usually the base is e. The principal scales C and D are graduated from 0.1 to 1 (or, if preferred, from 1 to 10). Instead, the scales A and B (scales of squares) contains a pair of consecutive intervals graduated from 0.1 to 1. Indeed, as a consequence of (5), $X = \log_b x$ yields $\log_b(x^2) = 2X$. Of course, the scale K of cubes contains three consecutive intervals graduated from 0.1 to 1.

Occasionally, other scales may be present, supplying the values of $1/x$, $\sqrt{1-x^2}$ etc., x being read on the scale D. The values of some trigonometric functions are available on the scale S,T,P. Sometimes, there are also scales which allow to compute directly percentages, and geometric quantities such as circumference and area of a circle of given radius.

There were also sliding rules designed for specific activity. The circular sliding rule of Figure 5 was a useful tool for installation of electrical devices such as generators, asynchronous motors and cable networks.

VIII. CURIOSITIES

• Sliding rules were constructed using different materials (wood, metal, plastic). The aforementioned form (a pair of sticks) is the typical one, but sliding rules have been realized also with the form of a disc and of a cylinder.

• The famous science fiction writer I. Asimov published in 1965 a handbook to explain how to use a sliding rule [1].

Fig. 5. Circular sliding rule for electrical installations

• The crew of mission *Apollo 11*, first humans landed on the Moon on July 20, 1969 (and other subsequent Moon missions) were equipped by sliding rules.

• The "Istituto Geografico Militare" of Firenze, location of this conference, published in 1889 a photographic reproduction of the *Thesaurus logarithmorum completus*, by G. Vega [7], one of the most influential books of logarithmic tables of ancient times.

• The already mentioned Vlacq's logarithmic tables [8] originally published in 1628, contained about 2100000 printed figures. Subsequent revisions performed with more efficient methods showed that 603 of this figures were wrong. The error percentage is less than 0.003%! This is really surprising, taking into account that all the computations, at that time, were made by hand.

• During the French revolution, Gaspard de Prony was committed to reorganizing the general real estate registry of France. In the context of this activity, he promoted a new edition of logarithmic tables, today known as *Tables du Cadastre*. The work was distributed at several levels of competence. To the lower level, only execution of sums was required. Many hair-dressers unemployed because of the decreasing demand of powdered wigs (plausibly in consequence of the revolution), were recruited for this job (for more details about this anecdote, see [3], p. 483).

IX. CONCLUSIONS

When pocket electronics calculators, around 1970, become popular and low cost, the sliding rule was rapidly abandoned and forgotten. The advantages of an electronic calculator are so many that the competition was unsustainable. However, it should be noticed that even by an electronic computer there are physical limits: for very large numbers, the result of a computation will be inevitably affected by errors. An electronic computer may guarantee more precision than the sliding rule, but not an absolute precision.

The moral of this tale points out that motivations for developing new mathematical tools are often related to reasons of high social interest. The same conclusion extends


TEORICA E PRATICA
DEL
REGOLO CALCOLATORE
PER
QUINTINO SELLA
Da veniam scriptis, quorum non gloria nobis
Causa, sed utilitas officiumque fuit.
Ovid. Fast. III, 9.
TORINO
STAMPERIA REALE
1859

Fig. 6. Original edition of Quintino Sella's book

to the design of instruments for computing of various nature (mechanical or electronic) and, more generally, to any scientific device for measuring and experimental research. The author apologizes for possible lack of historical precision and originality, which are beyond the purposes of this note.

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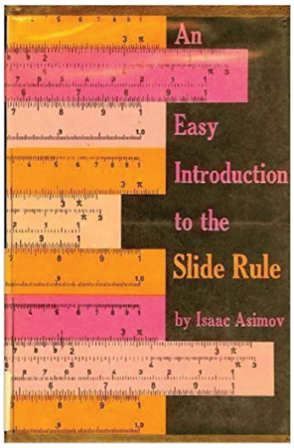

Fig. 7. Asimov's handbook