

A Novel Mathematical Structural Model Approach for Low Cost Structural Health Monitoring System

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Abstract –Italy is one of the Nation with the wider number of cultural heritage buildings and the monitoring of their health is the only way to preserve them. The integration of traditional Structural Health Monitoring (SHM) system in historical buildings is often very difficult due to the raw material employed, the geometric complexity of the constructions and the need to monitor multiple control points. Aim of the paper is to extend the monitoring of masonry historical constructions by using an innovative approach based on the use of finite elements mathematical models to detect a reduced number of points to monitor and establish the maximum stress that they can bear before to consider the whole structure in danger. Then an IoT based SHM for the online evaluation of the control point stress is proposed.

I. INTRODUCTION

Italy is often characterized by a historical cultural heritage damaged by several earthquakes that were occurred in the time. For example, on December 28, 1908, the deadliest earthquake ever occurred in Italy destroyed the cities of Messina, Sicily, and Reggio Calabria. This earthquake was characterized by an estimated moment of magnitude 7.1 and caused over 100,000 victims. The two main cities were totally destroyed, while the structures of the nearby towns collapsed partially, except those built with a particular cylindrical hollow clay brick identified with the name of fictile tubules [1]-[3].

Excluding cases of particular restoration and structural rehabilitation, the historical buildings are often subject to high seismic vulnerability and collapse. The structural health conditions of these constructions and the detection of damages and cracks are possible with the

implementation of the Structural Health Monitoring System (SHM) [4]-[7]. An evolution of SHM has been achieved in recent years thanks to the employment of the remote monitoring systems [8]-[9] and more recently the Internet of Things (IoT) paradigm [10]-[13]. The intervention in the management of cultural heritage through the integration of a SHM IoT allows the maintenance and management of structures in remote way in a control room, intervening on buildings only if strictly necessary, preventing damage and collapse of the structures while saving money [14]-[15].

Cultural heritage is a precious asset that must be protected and safeguarded. In order to guarantee this, it is necessary to have SHM systems capable of monitoring in real time the structural conditions of ancient buildings, providing prognostic and diagnostic data in real time, guaranteeing condition-based maintenance (CBM) and signaling the collapses before they occur avoiding economic expenditure [16]-[18].

To this aim, in the paper the cost of the monitoring system together with the peculiar needs of the monitoring of ancient structures are taken into account in order to propose a valid low cost solution easily applicable to a wide kind of historical structures. The peculiarities highlighted by the SHM systems used to date are related to the structure and type of SHM chosen. The particular and irregular geometries of masonry structures often make the analyzes conducted with commercial software inaccurate. It is due to too much simplification of the structure they require or, in the case of SHM based on images, to the artifacts affecting the acquired images [19]-[20]. A further problem is the material and the presence of barrel vaults and domes that are inserted only as loads without taken into account their collapse over time. The SHM, on the

other hand, can be very costly both for installation and maintenance.

In recent years several types of structural monitoring for masonry constructions have been proposed. A particular type is based on the acquisition of the data which must be used as input within the analyzes carried out on finite element numerical models. The main limitation is that these analyzes require a series of sensors located in several points of the structure and a high computational burden occurs and a dynamic FEM analysis can take up to 10 hours. Further limit is the cost of the monitoring system that has to manage and maintain several and heterogeneous sensors. In order to overcome such limits, in this paper a low cost SHM that needs a reduced number of sensors, all them accelerometers, and as a consequence a reduced number of data to process, is proposed. It foreseen a preliminary FEM analysis, that must be executed just once, to detect few monitoring points and the stress limit that such points must not overcome in order to consider the whole structure safe. Then the SHM acquire data from the accelerometers connected to an inverted pendulum, placed in the control point of the structure. These sensors will be able to provide information on dynamic stresses. The acquired data will be transmitted via the Internet to a computer, located in a control room, that will have the task of processing them online by using a suitable software based on a mathematical model. The main advantages provided by the mathematical model are two: the reduced computational burden, which allows to have information in a few seconds and the ease of implementation, being the inverted pendulum a common dynamic unstable model. The information obtained after the numerical analysis will be: the displacement of the control point, the maximum vibration angle of the pendulum and the force that generates it. All the data obtained will then be compared with the health limit values achieved and archived after the FEM analysis of the structure performed in the preliminary step.

Such computation can be executed in online mode, guarantying the timely intervention of maintainers or alarm generation.

II. SHM AND IOT

Structural Health Monitoring (SHM) aims to give, during the life of a civil engineering structures, a diagnosis of the “state” of the raw materials, of the different parts, and of the full assembly of the structures under monitoring.

The state of the structure must remain in the safety domain specified by the engineers, but often the normal aging or the action of meteorological agents, accidents or other cause of stress can alter it, suddenly or during time. Thanks to the continuous monitoring during time, it is possible to consider the full history of the structure, and with the help of “Usage Monitoring” (UM) techniques, it is possible to provide a prognosis (evolution of cracks, localization of damage, residual life, etc.). In light of this, it is possible to estimate that

SHM is an useful tools that should be applied in every building and structures in order to save human life, preserve the cultural heritage, reduce the cost of maintenance. SHMs implemented with the paradigm of Internet of Things (IoT) [21]-[28] are much more. Indeed, these systems involve the integration of sensors, smart materials, data transmission, computational power, and processing ability. On the basis of the achieved data, that should managed according to the Big Data paradigm and the Data Retrieval techniques, it is possible to reconsider the design of the new structures and the full management of the existent structures by considering all structures as a part of wider system monitored, for example, in online mode.

IoT smart objects are used to collect data in the monitored structures with the goal of improving the safety trough a deeper knowledge of stress in the material and then allows the action of expert to reduce the development of the damage [29].

Usually SHM based on IoT are characterized by a distributed system based on battery powered low cost devices able to measure, for example, acoustic emissions [31]-[34] and strain in the material [3]. A system of this kind applies the IoT paradigm in the monitoring scenario of the structural elements like beams or pillars, briefly considering the reference communication stack [35].

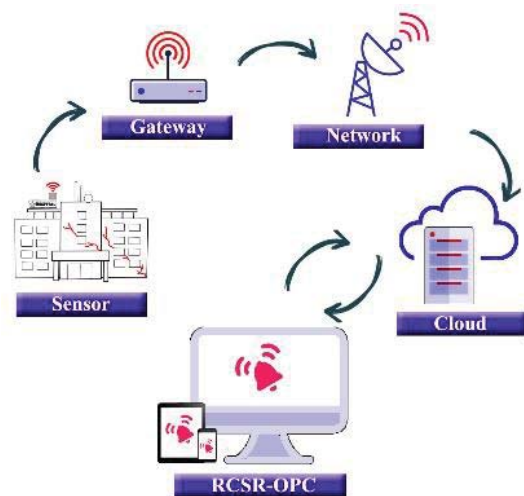


Fig. 1. Example of a IoT scheme applicate to SHM [11].

All monitoring devices (which in this case are intelligent IoT objects), have to guarantee the acquisition of measurement information to identify model variables and detected events. Consequentially, each resource can be managed using a protocol and paradigm that bling in the constrained environment. From an architectural point of view, an IoT based SHM is similar to typical traditional structural monitoring solutions, but with significant differences in the role of its components. In fact, they are not simple devices but real IoT smart objects with the ability to process data and send only synthetic high level measurement information. The

architecture of these system is commonly based on four principal components: (i) smart sensor devices, (ii) gateway, (iii) remote control and Service Room (RCSR), and (iv) Open Platform Communications (OPC) server as shown in Figure 1.

III. THE PROPOSED SHM SYSTEM

The preliminary SHM proposed in this article is designed with the aim of being able to monitor the dynamic stresses acting on historical structures while assessing a fast evaluation of the damage and a low cost. As said before the proposed SHM is based on:

- i) The preliminar analysis of the monitored structure with a FEM model in order to detect a reduced number of monitoring points.
- ii) The design of the Smart IoT SHM node,
- iii) The software to elaborate the information measurements acquired by the Smart IoT SHM nodes.

A. Analysis of the Monitored Structures with FEM Model

To date, in the field of structural engineering there are several methods for proceeding with the seismic analysis of a masonry building. Some of them are common and use commercial software based on the equivalent frame method, others, are more complex and require a higher computational burden [36]-[38]. For regular masonry constructions, which can be simplified by means of piers and spandrels, the equivalent frame method is fast and also reliable. When this is not possible - as in the case of particular historical monumental buildings - continuous modeling is necessary because it reproduces, more faithfully, the behavior under seismic action [39]-[41]. In the light of this, approaches based on FEM are more suitable because they allow to model structures having any geometry. They associate sophisticated models with the constituent materials and are capable of taking into account the strongly non-linear behavior of the masonry. The continuous calculation models are discretized by means of a mesh of tetrahedral finite elements which do not represent the individual blocks of brick but the structure through specific criteria and user-defined constitutive laws.

There are two main analyzes that can be carried out on a structure to understand its seismic vulnerability: nonlinear static analysis (pushover) and nonlinear dynamic analysis.

Static non-linear analysis generally takes place by subjecting the structure step by step to forces - growing monotonously - until it collapses. The structural response will evolve over time - without however assuming a physical meaning as it will essentially coincide with the duration of the simulation - gradually bringing the elements of the structure to collapse. Consequently, at each event, the structure will undergo a loss of stiffness [42]-[43] and the analysis can be conducted in control of forces or displacement. In non-linear dynamic analysis, the structure is subjected, step by step, to loads and the structural response evolves in time, also taking

into account the effects of damping and inertia. The effects of the dynamic actions induced by the earthquake or other stress events on the wall structures can be assessed, in a nonlinear dynamic analysis, through an artificial accelerogram compatible with the elastic spectrum, consistent with the response spectrum defined by the legislation, along the two orthogonal horizontal directions. For the execution of the analysis, explicit or implicit time integration schemes can be used to satisfy the equations of motion during each time phase of the simulation [44]-[46]. In an explicit method, the new response values calculated in each step are based only on the quantities calculated in the previous step. On the contrary, in an implicit method, the expressions that generate the new values for a given step also include values that belong to the same step. In both analyzes, the aim is to identify the maximum displacement of the control points and how it is correlated with the state of stress that occurs within the masonry.

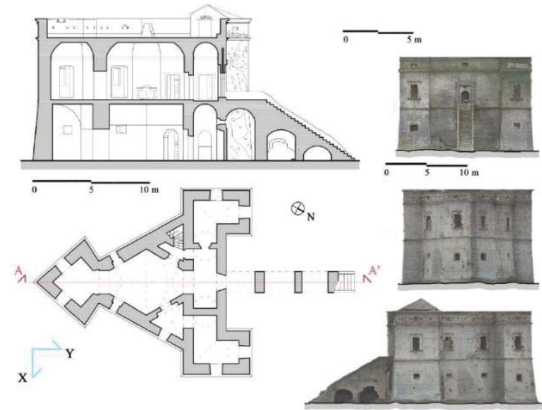


Fig. 2. Example of a IoT scheme applicate to SHM

The numerical approach allows also to identify the phases to appreciate a preliminary SHM realized through an implementation of the mathematical model in Simulink. The steps typically developed are two: i) identification of the control points and health limit values of the structure ii) calibration of the mathematical model of the inverse pendulum used to assess the stress of the specific monitoring point of the structure.

The first step was carried out taking into consideration a non-regular structure located in the territory of southern Calabria, the Castle of San Fili, which was subjected of an accurate architectural survey (Figure 2 [47]).

A FEM model of the structure was subsequently implemented in ABAQUS and non-linear static and non-linear dynamic analyzes were conducted in order to identify the weak points of the structure and the damage mechanisms that are generated (Figure 3). In this way, the control points to be monitored and the capacity curves of the structure were identified. In particular, in the structure showed in Fig.3, the FEM analysis highlights only 3 points to be monitored, in the picture they are denoted with C1, C2, C3. This also made possible to recognize the damage limit values to be compared with those obtained by the monitoring system.

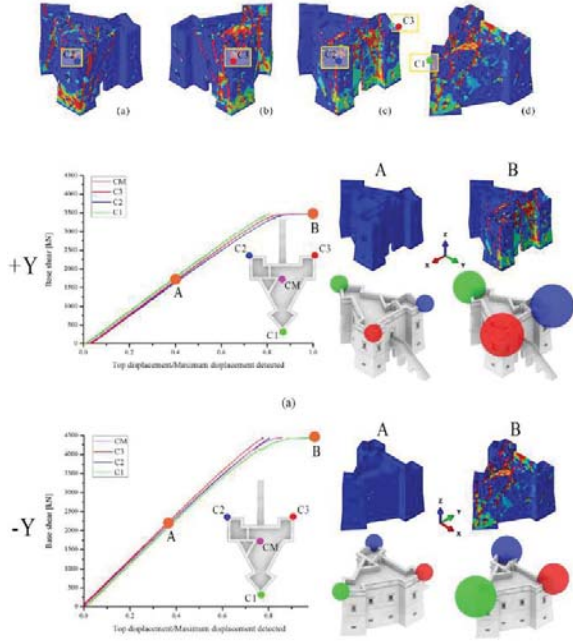


Fig. 3. Control points and Load vs Displacement curves obtained in Y direction

All the results obtained in terms of maximum displacement of the control points, applied force and load vs displacement curves, have been recorded and stored in a cloud in order to be controlled with the data obtained by the monitoring system.

- B. The smart node of the proposed SHM IoT Based and the Novel Mathematical Model to evaluate the displacement

According to the mathematical model described in subsection III.A, the IoT smart node is constituted by an inverse pendulum equipped with accelerometers.

A mathematical model of an inverted pendulum was created using Simulink, a Matlab add-in. This basic model of the mathematical physics has been optimized to guarantee a better analysis of the data acquired in the future. In particular, in comparison to the analyzes in literature, the study of the value of the friction between support and cart has been varied

The inverted Pendulum, schematized in figure 4 is a typical unstable nonlinear system governed by the following second order differential equations [48]-[50]:

$$\frac{d^2x}{dt^2} = \frac{1}{m} \left(F - N - b \frac{dx}{dt} \right) \quad (1)$$

$$\frac{d^2\theta}{dt^2} = \frac{1}{I} (NL \cdot \cos(\theta) + PL \cdot \sin(\theta)) \quad (2)$$

Where F is the force that induced the displacement x , b is the friction of the cart imposed equal to 1.2 in order to simulate the interlocking constraint, I is the inertia of pendulum, L is the length to pendulum centre of mass, P and

N are respectively the horizontal and vertical reaction to the applied force F and ϑ is the inclination angle of the pendulum.

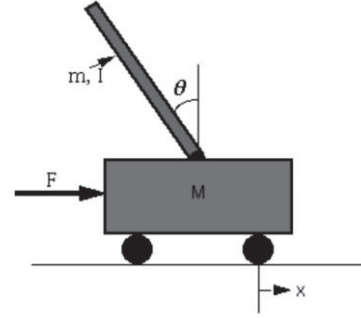


Fig. 4. Diagram of an Idealized Pendulum in its Inverted State

The smart nodes are installed in the control points of the structure, identified in step 1. This will be done in order to obtain the oscillation of the pendulum and to achieve all the control data of the monitoring system, reducing the computational burden of the analysis.

The archived information, such as the maximum displacement of the control point, are transmitted through internet to the Service Room where are compared, in online modality, with the health limit values obtained by the FEM model. Such a comparison allows to determine the damage occurred during the earthquake. This guarantees a quick analysis and evaluation of the structural stability of the construction.

IV. NUMERICAL VALIDATION OF THE MATHEMATICAL MODEL IMPLEMENTED IN THE SMART NODE

In order to obtain a first validation of the proposed mathematical model, numerical tests were performed with the aim to compare its result with the behavior of the ones presented in [44] by numerical simulation. To this aim, in the proposed model the experimental conditions of [51], in part, are imposed: mass of the cart (M): 0.5 kg, mass of the pendulum (m): 0.2 kg, length of the pendulum (L): 0.6 m. The friction coefficient of the cart is not considered negligible but equal to 1.2 in order to simulate the interlocking constrain. A pulse that stresses the equilibrium of the system by inducing a displacement of the cart is given.

In particular, the pulse duration is 0.01 s, the amplitude is 50 N. The first results obtained in figures 5 and 6 show the movement and oscillation of the inverted pendulum, that are compatible with the trend of the results in [51]. The applied force F has induced a 0.041 m displacement of the cart and an oscillation of the pendulum with a maximum angle of 21.19° . The analysis of the dynamic system in an interval of 20 seconds has shown how the equilibrium condition after the application of F is restored both as concerning the displacement of the cart (Fig.5) and the oscillation of the pendulum (Fig. 6) and that the data obtained are also compatible with the ones arising from the FEM analysis being both of the same order of magnitude.

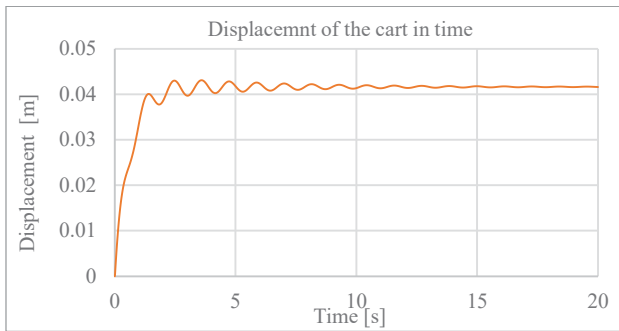


Fig. 5. Displacement of the cart in time

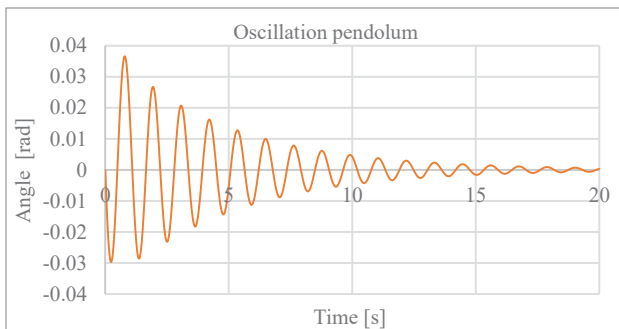


Fig. 6. Oscillation of the pendulum

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