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PERFORMANCE FUNCTION FOR CASCADED RECTANGULAR FILTERS IN SPECTRUM ANALYSIS

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Abstract – A different procedure, introduced previously by the authors for evaluating the errors due to the equally spaced sampling technique in the spectral analysis of a periodic signal and applied to the rectangular window, has been used to evaluate the effect of the cascade of two and three rectangular windows. This procedure is based on the assumption that the initial sampling instant can be randomized so that each estimated spectral component can be considered a random variable characterized by its statistical parameters. The theoretical expressions of these parameters were deduced and compared with those of the rectangular window; beside the theoretical findings were compared with the simulated ones denoting a very good agreement between them.

Key words: equally spaced sampling, performance function, error analysis, statistical parameters, windowing.

1. INTRODUCTION

The properties of digital spectral analysis based on the equally-spaced sampling procedure has been widely studied through the discrete-time Fourier Transform and the discrete Fourier Transform [1, 2, 3]; recently [4] we introduced a different method for deducing the errors arising from the sampling strategy and the filtering procedure. To this end we observed that an estimator of the spectral components of a periodic input signal is characterised by measurement errors which depend on the number of samples considered and shows an intrinsic variability related to the initial sampling instant, which is unknown. By assuming this instant, according to the Bayesian approach, as a random variable with a distribution function correlated to the prior information available, the estimator of each spectral component of the periodic input signal becomes a random function, and its variability can be expressed through the well-known statistical parameters, i.e. mean value and mean square error. In order to deduce the properties of each estimator is therefore necessary to find the expressions of these parameters. Previously we deduced them when a rectangular window is used to approximate the integral of the Fourier series; the purpose of this paper is to deduce their expressions when the cascade of two or three rectangular windows is considered and to compare these windows with the rectangular one. Besides the theoretical findings were compared with the simulated ones.

2. MEASUREMENT PROCEDURE

A real signal of period $T_1 = 1/f_1$ can be expressed through the Fourier series as follows:

$$x(t) = \sum_{q=-M}^{+M} X_q \mathrm{e}^{\mathrm{j}2\pi q f_1 t} \tag{1}$$

with $X_{-q} = X_q^*$ and X_0 real, where a generic spectral components X_n of $x(t)$ can be deduced by this expression:

$$X_n = \frac{1}{T_1} \int_{-T_1/2}^{+T_1/2} x(t) \mathrm{e}^{-\mathrm{j}2\pi n f_1 t} \mathrm{d}t \tag{2}$$

In order to estimate each of these spectral components X_n through an equally spaced sampling sequence of N signal values in correspondence of successive equal intervals T_s, beginning from the initial instant t_I, it can be used the following formula:

$$\widetilde{X}_n = \frac{1}{N} \sum_{k=1}^{N} x(t_k) \mathrm{e}^{-\mathrm{j}2\pi n f_1 t_k} = \sum_{k=1}^{N} w[k] x(t_k) \mathrm{e}^{-\mathrm{j}2\pi n f_1 t_k} \tag{3}$$

with $\widetilde{X}_{-n} = \widetilde{X}_n$, where:

$$t_k = t_I + kT_s \tag{4}$$

is the sampling instant and

$$w[k] = \begin{cases} \dfrac{1}{N} & \text{for} \quad k = 1...N \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

with $\sum_{k=1}^{N} w[k] = 1$ is the sequence of a discrete rectangular window, which it is known also as the Dirichlet window [5]. Obviously, both the spectral component X_n and its estimator $\widetilde{X}_n$ are complex quantities. By substituting eqn.1 and 4 into eqn.3 it results:

$$\widetilde{X}_n = \sum_{k=1}^{N_1} w[k] \sum_{q=-M}^{+M} X_q \mathrm{e}^{-\mathrm{j}2\pi(q-n) f_1 (t_I + kT_s)} =$$

$$= X_n + \sum_{\substack{q=-M \\ q\neq n}}^{+M} X_q \mathrm{e}^{-\mathrm{j}2\pi(q-n)f_1 t_i} W((q-n)f_1 T_s) \qquad (6)$$

where, in the last passage it has been imposed $\sum_{k=1}^{N} w[k]=1$ in order to obtain an unbiased estimator of the considered spectral component. On the hypothesis of a rectangular window (eqn.5) the function $W(fT_s)$, which is the Fourier Transform of the real window sequence, results:

$$W(f)=\sum_{k=1}^{N} w[k]\mathrm{e}^{\mathrm{j}2\pi fkT_s} = \mathrm{e}^{\mathrm{j}\pi(N+1)fT_s}\frac{\mathrm{sinc}(NfT_s)}{\mathrm{sinc}(fT_s)} \qquad (7)$$

with $f=(q-n)f_1$. In order to reduce the spectral leakage associated with a finite observation interval the cascade of two rectangular windows, the first with N_1 samples and the second one with N_2 samples, can be used according to the following formula:

$$\widetilde{X}_n^{(2)} = \frac{1}{N_1}\sum_{p=1}^{N_1}\frac{1}{N_2}\sum_{k=p}^{p+N_2-1} x(t_k)\mathrm{e}^{-\mathrm{j}2\pi n f_1 t_k} \qquad (8)$$

By introducing two rectangular windows $w_1[k]$ and $w_2[k]$ defined according to eqn.5 respectively for $N=N_1$ and $N=N_2$, eqn.8 can be rewritten as follows:

$$\begin{aligned}\widetilde{X}_n^{(2)} &= \sum_{p=1}^{N_1} w_1[p]\sum_{k=p}^{p+N_2-1} w_2[k+1-p]x(t_k)\mathrm{e}^{-\mathrm{j}2\pi n f_1 t_k} = \\ &= \sum_{k=-\infty}^{+\infty} w^{(2)}[k]x(t_{k-1})\mathrm{e}^{-\mathrm{j}2\pi n f_1 t_{k-1}}\end{aligned} \qquad (9)$$

where:

$$w^{(2)}[k]=\sum_{p=-\infty}^{+\infty} w_1[p]w_2[k-p] = w_1[k] * w_2[k] \qquad (10)$$

is the window sequence of the cascade of two rectangular windows of N_1 and N_2 values, respectively of $1/N_1$ and $1/N_2$ amplitude, and it can be deduced as the discrete convolution of the above said two rectangular windows. By recalling again that the discrete rectangular functions $w_1[p]$ and $w_2[k-p]$ are defined according to eqn.5, i.e. the sequence of the not null values are of amplitude respectively $1/N_1$ for $1\le p\le N_1$ and $1/N_2$ for $1\le k-p\le N_2$ (consequently $k-N_2\le p\le k-1$). By substituting the extrema of p in this last inequality, we obtain that it must be verified both the conditions $1\le k-1$ and $k-N_2\le N_1$. Therefore $w^{(2)}[k]$ is not null in the range $2\le k\le N_1+N_2$. Taking into account the conditions which the variable p must satisfy, eqn.10 can be rewritten as follows:

$$w^{(2)}[k]=\sum_{r=\max(1,k-N_2)}^{\min(N_1,k-1)} w_1[r]w_2[k-r] \quad \text{for } 2\le k\le N_1+N_2 \qquad (11)$$

with $N_1>N_2$ and zero otherwise. Therefore $w^{(2)}[k]$ is the sequence of a trapezoidal window of N_1+N_2-1 samples defined as follows:

$$w^{(2)}[k]=\begin{cases}\dfrac{k-1}{N_1N_2} & \text{for } 2\le k\le N_2 \\ \dfrac{1}{N_1} & \text{for } N_2<k\le N_1 \\ \dfrac{N_1+N_2-(k-1)}{N_1N_2} & \text{for } N_1<k\le N_1+N_2\end{cases} \qquad (12)$$

on the hypothesis that $N_1>N_2$; otherwise it is necessary to invert N_1 with N_2. It can be shown that $\sum_{k=2}^{N_1+N_2} w^{(2)}[k]=1$. Because the number of samples is N_1+N_2-1, the trapezoidal window must be compared with a rectangular window which has the same number of samples. When $N_1=N_2=N$ the second relationship into eqn.12 does not exist; the correspondent window becomes triangular and it is known also as Bartlett window [5]. The function has a maximum for $k=N+1$, whose value is $w^{(2)}[N+1]=\frac{1}{N}$, and it is symmetric with respect to this maximum.

By substituting eqn.1 and 4 into eqn.9 it results:

$$\widetilde{X}_n^{(2)} = \sum_{k=-\infty}^{+\infty} w^{(2)}[k]\sum_{\substack{q=-M \\ q\neq 0}}^{+M} X_q \mathrm{e}^{\mathrm{j}2\pi(q-n)f_1(t_i+(k-1)T_s)} =$$

$$= X_n + \sum_{\substack{q=-M \\ q\neq 0 \\ q\neq n}}^{+M} X_q \mathrm{e}^{\mathrm{j}2\pi(q-n)f_1(t_i-T_s)} W^{(2)}((q-n)f_1T_s) \qquad (13)$$

where, by recalling eqn.10 and assuming $f=(q-n)f_1$, it follows:

$$\begin{aligned}W^{(2)}(fT_s) &= \sum_{k=-\infty}^{+\infty} w^2[k]\mathrm{e}^{\mathrm{j}2\pi fkT_s} = \\ &= \sum_{k=-\infty}^{+\infty}\sum_{p=-\infty}^{+\infty} w_1[p]w_2[k-p]\mathrm{e}^{\mathrm{j}2\pi fkT_s} = \\ &= W_1(fT_s)W_2(fT_s)\end{aligned} \qquad (14)$$

In the third passage it has been assumed $k-p=p'$ while in the last passage it has been taken into account eqn.7 with two different numbers of the values of the rectangular sequence, respectively N_1 and N_2. Therefore the weighting function of the cascade of two rectangular windows can be obtained as the product of the weighting functions of each of the two rectangular windows, as it is obvious.

From eqn.6 the expression for the cascade of three rectangular filters, respectively with N_1, N_2 and N_3 samples, can be immediately deduced by using the same criterion:

$$\tilde{X}_n^{(3)} = \frac{1}{N_1}\sum_{r=1}^{N_1} \frac{1}{N_2}\sum_{p=r}^{r+N_2-1} \frac{1}{N_3}\sum_{k=p}^{p+N_3-1} x(t_k)\mathrm{e}^{-\mathrm{j}2\pi n f_1 t_k} \qquad (15)$$

By introducing three rectangular windows $w_1[k]$, $w_2[k]$ and $w_3[k]$ defined according to eqn.5 respectively for $N = N_1$, $N = N_2$ and $N = N_3$, eqn.15 can be rewritten as follows:

$$\begin{aligned} \tilde{X}_n^{(3)} &= \sum_{r=1}^{N_1} w_1[r] \sum_{p=r}^{r+N_2-1} w_2[p+1-r] \cdot \\ &\quad \cdot \sum_{k=p}^{p+N_3-1} w_3[k+1-p]x(t_k)\mathrm{e}^{-\mathrm{j}2\pi n f_1 t_k} = \\ &= \sum_{k=-\infty}^{+\infty} w^{(3)}[k]x(t_{k-2})\mathrm{e}^{-\mathrm{j}2\pi n f_1 t_{k-2}} \end{aligned} \qquad (16)$$

where:

$$\begin{aligned} w^{(3)}[k] &= \sum_{r=-\infty}^{+\infty} w_1[r] \sum_{p=-\infty}^{+\infty} w_2[p-r] w_2[k-p] = \\ &= w_1[k] * w_2[k] * w_3[k] \end{aligned} \qquad (17)$$

can be deduced as the discrete convolution of the three rectangular windows. On the hypothesis of three equal rectangular windows with $N_1 = N_2 = N_3 = N$ sequence values equal to $1/N$, the function $w^{(3)}[k]$ must satisfy the following relationship:

$$w^{(3)}[k] = \begin{cases} \displaystyle\sum_{p=\max(2,k-N)}^{\min(2N,k-1)} w^{(2)}[p] w_3[k-p] & \text{for } 3 \le k \le 3N \\ 0 & \text{otherwise} \end{cases} \qquad (18)$$

where from eqn.12 it follows:

$$w^{(2)}[p] = \begin{cases} \dfrac{p-1}{N^2} & \text{for } 2 \le p \le N \\ \dfrac{2N-(p-1)}{N^2} & \text{for } N < p \le 2N \end{cases} \qquad (19)$$

while $w_3[k-p]$ is defined according to eqn.5, i.e. the sequence values are $1/N$ for $1 \le k-p \le N$. By substituting the extrema of p in this last inequality, it must be verified both the conditions $2 \le k-1$ and $k-N \le 2N$; consequently we have $3 \le k \le 3N$. By imposing all these conditions, from eqn.18 it results:

$$w^{(3)}[k] = \begin{cases} \dfrac{1}{2N^3}(k-2)(k-1) & \text{for } 3 \le k \le N+1 \\ \dfrac{1}{2N^3}\left[-3N^2 - 9N - 4 + 6(N+1)k - 2k^2\right] & \text{for } N+1 < k \le 2N \\ \dfrac{1}{2N^3}\left[9N^2 + 9N + 2 - 3(2N+1)k + k^2\right] & \text{for } 2N < k \le 3N \end{cases} \qquad (20)$$

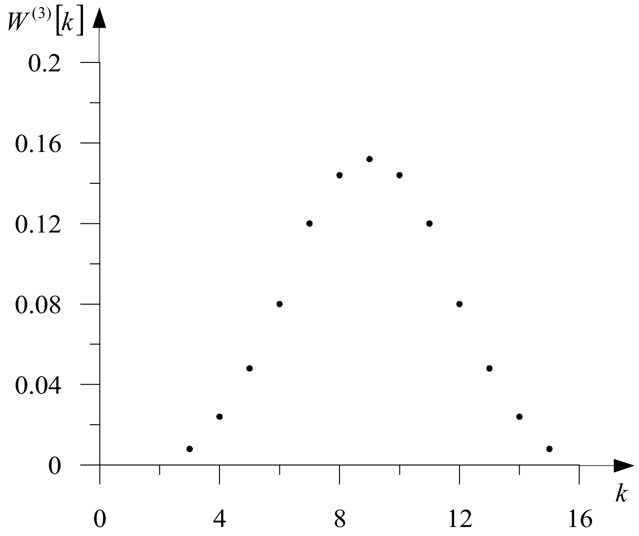

Fig. 1. Convolution of three equal rectangular windows with $N = 5$.

It can be shown that $\sum_{k=3}^{3N} w^{(3)}[k] = 1$. Because the number of samples is $3N-2$, this window must be compared with the correspondent rectangular window with an equal number of samples, i.e. $3N-2$. It can be shown that, for N odd, the function expressed by eqn.20 has a maximum for $k = \frac{3}{2}(N+1)$ equal to $w^{(3)}\left[\frac{3}{2}(N+1)\right] = \frac{3}{4N} + \frac{1}{4N^3}$. Figure 1 shows the shape of this window obtained as the convolution of three equal rectangular windows with $N = 5$.

By substituting eqn.1 and 4 into eqn.16 it results:

$$\begin{aligned} \tilde{X}_n^{(3)} &= \sum_{k=-\infty}^{+\infty} w^{(3)}[k] \sum_{q=-M}^{+M} X_q \mathrm{e}^{\mathrm{j}2\pi(q-n)f_1(t_I+(k-2)T_s)} = \\ &= X_n + \sum_{\substack{q=-M \\ q \ne n}}^{+M} X_q \mathrm{e}^{\mathrm{j}2\pi(q-n)f_1(t_I-2T_s)} W^{(3)}((q-n)f_1 T_s) \end{aligned} \qquad (21)$$

where, by recalling eqn.17, in analogy to eqn.14 it results:

$$W^{(3)}(fT_s) = W_1(fT_s)W_2(fT_s)W_3(fT_s) \qquad (22)$$

3. CRITERION FOR THE PERFORMANCE ANALYSIS

In the following only the errors arising from the sampling strategy and the filtering procedure are considered since this paper aims to deduce their specific properties in the spectrum analysis of a periodic signal. From eqn.6 it follows:

$$\tilde{X}_n(t_I) = X_n + \Delta_n(t_I) \qquad (23)$$

where:

$$\Delta_n(t_I) = \sum_{\substack{q=-M \\ q \ne n}}^{+M} X_q \mathrm{e}^{\mathrm{j}2\pi(q-n)f_1 t_I} W((q-n)f_1 T_s) \qquad (24)$$

is a complex quantity with $\Delta_{-n}(t_I) = \Delta_n^*(t_I)$; consequently also $\tilde{X}_{-n}(t_I) = \tilde{X}_n^*(t_I)$ because $X_{-n} = X_n^*$. The initial sampling instant t_I, being unknown, can be interpreted as a continuous random variable; therefore the measurement error $\Delta_n(t_I)$, introduced by the estimator $\tilde{X}_n(t_I)$ is random itself since it is a function of the random variable t_I. An appropriate characterization of the output uncertainty can be obtained by evaluating the statistical parameters of the estimator $\tilde{X}_n(t_I)$, i.e. the mean value and the mean square error. Since any prior information on t_I is missing, it can be assumed, according to the Bayesian approach, as a random variable with a uniform distribution in some generic time interval T [4,6]. In order to avoid the influence of the conventional time origin on the instrument performance, the excursion T of the initial shift t_I must be sufficiently large and theoretically must tend to infinity. Therefore we consider the asymptotic statistical parameters, i.e. the asymptotic mean and the asymptotic mean-square error.

It can be shown that the output of the instrument is asymptotically unbiased (see in Appendix eqn.A6):

$$\underset{\rightarrow}{\mathrm{E}}\{\tilde{X}_n(t_I)\} = X_n + \underset{\rightarrow}{\mathrm{E}}\{\Delta_n(t_I)\} = X_n \qquad (25)$$

since $\underset{\rightarrow}{\mathrm{E}}\{\Delta_n(t_I)\} = 0$ for eqn.A5. Consequently the asymptotic mean-square error coincides with the asymptotic variance $\underset{\rightarrow n}{\sigma}^{(2)2} = \underset{\rightarrow}{\mathrm{Var}}\{\tilde{X}_n(t_I)\} = \underset{\rightarrow}{\mathrm{Var}}\{\Delta_n^{(2)}(t_I)\}$. The final expression of this last equation is (see eqn.A10 in Appendix):

$$\underset{\rightarrow n}{\sigma}^{(1)2} = \underset{\rightarrow}{\mathrm{E}}\{\Delta_n(t_I)\Delta_n^*(t_I)\} = \sum_{\substack{q=-M \\ q\neq n}}^{+M} |X_q|^2 |W((q-n)f_1T_s)|^2 \qquad (26)$$

where, from eqn.7 we have:

$$|W(fT_s)|^2 = \frac{1}{N^2}\frac{\sin^2(\pi NfT_s)}{\sin^2(\pi fT_s)} = \frac{\mathrm{sinc}^2(NfT_s)}{\mathrm{sinc}^2(fT_s)} \qquad (27)$$

being $f = (q-n)f_1$. Equation 26, deduced with another procedure in a previous paper [6], shows that the asymptotic variance of the estimator of each n^{th} spectral component can be deduced by summing the squared amplitudes of the remaining spectral components each multiplied by the product of the weighting function defined by the eqn.27, and evaluated at the frequencies $(q-n)f_1T_s$. Because the generic spectral component of a real signal is represented by the sum of two periodic exponentials which are the complex coniugate of each one, from eqn.26 we deduce that also the other component with respect to the considered one contributes to the variance.

The obtained weighting function is periodic of unit period as a function of fT_s; within the first period $(0 < fT_s < 1)$, it assumes $(N-1)$ zeros for:

$$Nf_{i0}T_s = i \qquad \text{for } i = 1,....,N\text{-}1 \qquad (28)$$

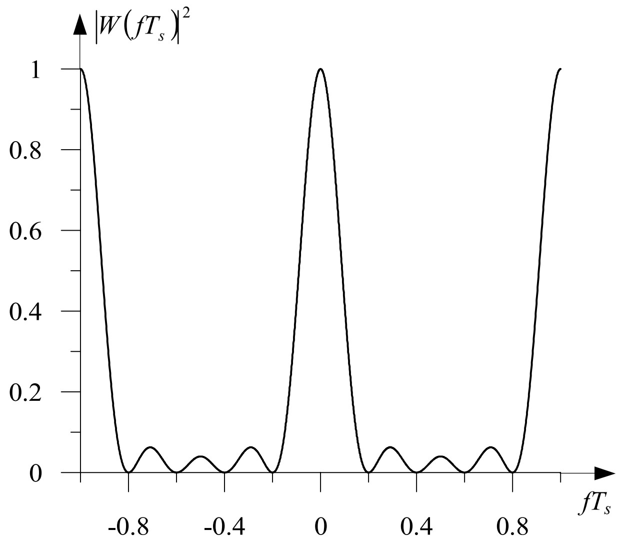

Fig. 2. Weighting function of eqn.27 with $N = 5$.

and $(N-2)$ central values between two successive zeros for [6]:

$$f_{ic}T_s = f_{i0}T_s + \frac{1}{2N} = \frac{1}{N}\left(i + \frac{1}{2}\right) \qquad \text{for } i = 1,......,N\text{-}2 \qquad (29)$$

in the first period. The values of these central values (which represent the maximum of each side-lobe) are given by:

$$|W(f_{i\max}T_s)|^2 = \frac{1}{N^2}\frac{1}{\sin^2\left(\frac{2i+1}{N}\frac{\pi}{2}\right)} \qquad (30)$$

In order to reduce the amplitude of the side-lobes it is necessary to impose an adequate value to N. It is interesting to observe that, on the hypothesis of an odd number of samples, for $fT_s = 1/2$ it can be deduced immediately from eqn.27 that $W^2(0.5) = 1/N^2$; therefore it is convenient to increase the value of N in order to reduce the maxima of the sidelobes. Figure 2 shows the shape of the weighting function of eqn.27 for $N = 5$, with the main lobes around the values of fT_s integers and the side-lobes. In the asyncronous sampling, i.e. $NT_s \neq rT_1$, the frequency of the input signal is not related to the sampling interval; therefore it can be assume any value. In order to avoid that the component nearest to the considered one be within the main-lobe, it is necessary to impose that $f_1T_s > 1/N$, i.e. a windows length greater than the period of the input signal; this condition represents the resolution of the instrument. Due to the sidelobes, all the other spectral components of the signal contribute to the variance relative to any particular spectral component; this effect is known as the spectral leakage. By considering the asymptotic variance of the greatest spectral component (M^{th}), we deduce that the maximum argument of the weighting function corresponds to $2Mf_1T_s$ (i.e. for $q = M$ and $n = -M$, or the opposite); in order to avoid values of the weighting function near the unity, the condition $2Mf_1T_s < 1$ must be imposed, according to the sampling theorem. However, it is more convenient to impose, also in this case, a more restrictive condition in order to avoid the main lobe correspondent to $fT_s = \pm 1$; to this end it is necessary that $2Mf_1 < f_s - f_s / N$. On the hypothesis instead of a synchronous sampling, i.e. $NT_s = rT_1$ with N necessarily odd, the frequency leakage is null because the weighting function $W(f)$ for $f = (q-n)f_1$ is null coinciding with its zeroes. Besides the condition relative to the resolution is automatically satisfied. By imposing the condition of the synchronous sampling in the weighting function, this last results equal to one only for $(q-n) = kN$, with k an integer, and equal to zero otherwise. Therefore the sampling theorem must be expressed as follows: $2Mf_1 < Nf_1 = bf_s$, i.e. the maximum frequency depends on the number of period of the input signal in which the N samples are taken.

By using the same procedure when we consider the cascade of two rectangular filters, from eqn.13 and 14 we deduce:

$$\tilde{X}_n(t_I) = X_n + \Delta_n(t_I) \qquad (31)$$

where:

$$\Delta_n^{(2)}(t_I) = \\ = \sum_{\substack{q=-M \\ q\neq n}}^{+M} X_q \mathrm{e}^{\mathrm{j}2\pi(q-n)f_1(t_I - T_s)} W_1((q-n)f_1T_s) W_2((q-n)f_1T_s) \qquad (32)$$

This equation is similar to eqn.24 and consequently the asymptotic statistical parameters of the estimator can be deduced by eqns.25 and 26 as follows:

$$\underset{\rightarrow}{\mathrm{E}}\{\tilde{X}_n(t_I)\} = X_n + \underset{\rightarrow}{\mathrm{E}}\{\Delta_n(t_I)\} = X_n \qquad (33)$$

$$\underset{\rightarrow n}{\sigma}^{(2)2} = \underset{\rightarrow}{E}\{\Delta_n^{(2)}(t_I)\Delta_n^{*(2)}(t_I)\} = \\ = \sum_{\substack{q=-M \\ q\neq n}}^{+M} |X_q|^2 |W_1((q-n)f_1T_s)|^2 |W_2((q-n)f_1T_s)|^2 \qquad (34)$$

Similarly, when we introduce the cascade of three rectangular windows, by referring to eqns.21 and 22, we obtain:

$$\underset{\rightarrow}{\mathrm{E}}\{\tilde{X}_n^{(3)}(t_I)\} = X_n + \underset{\rightarrow}{\mathrm{E}}\{\Delta_n^{(3)}(t_I)\} = X_n \qquad (35)$$

$$\underset{\rightarrow n}{\sigma}^{(3)2} = \underset{\rightarrow}{E}\{\Delta_n^{(3)}(t_I)\Delta_n^{*(3)}(t_I)\} = \\ = \sum_{\substack{q=-M \\ q\neq n}}^{+M} |X_q|^2 |W_1((q-n)f_1T_s)|^2 \cdot \qquad (36) \\ \cdot |W_2((q-n)f_1T_s)|^2 |W_3((q-n)f_1T_s)|^2$$

The comparison between the rectangular window and the cascade of two or three rectangular windows in an asynchronous sampling, i.e. $NT_s \neq rT_1$, can be made by considering an equal number of samples. The continuous line in fig.3 shows the shape of the weighting function $|W(fT_s)|^2$ given by eqn.27 as a function of fT_s for $N = 7$,

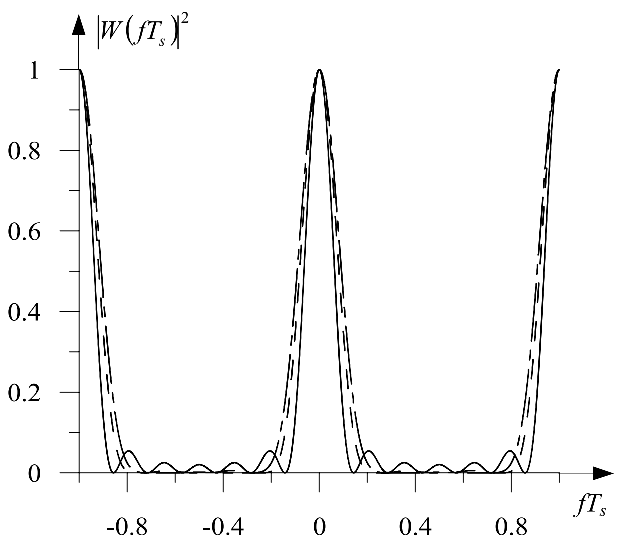

Fig. 3. Comparison between the three proposed windows on the hypothesis of the same number of samples.

while the dashed line shows the shape of the product of the two weighting functions $|W_1(fT_s)|^2 |W_2(fT_s)|^2$ as a function of fT_s for $N_1 = N_2 = N' = 4$ (therefore the condition $N = N_1 + N_2 - 1$ is verified) and the dashdot shows the shape of the product of the three weighting functions $|W_1(fT_s)|^2 |W_2(fT_s)|^2 |W_3(fT_s)|^2$ as a function of fT_s for $N_1 = N_2 = N_3 = N'' = 3$ (therefore the condition $N = N_1 + N_2 + N_3 - 2$ is verified). From the comparison of the three shape it can be deduced that the resolution, due to the main lobe, decreases lightly with the order of the cascade of the rectangular filters, while the frequency leakage, due to the sidelobes, strongly decreases. Because the main effect of the synchronous sampling strategy is to null the frequency leakage, the cascade of rectangular windows is not convenient in this case.

By recalling that it results [4]:

$$\underset{\rightarrow}{\mathrm{E}}\{\tilde{X}_n(t_I)\} = \underset{\rightarrow}{\mathrm{E}}\{\mathrm{Re}[\tilde{X}_n(t_I)]\} + \mathrm{E}\{\mathrm{Im}[\tilde{X}_n(t_I)]\} \qquad (37)$$

$$\underset{\rightarrow}{\mathrm{Var}}\{\tilde{X}_n(t_I)\} = \underset{\rightarrow}{\mathrm{Var}}\{\mathrm{Re}[\tilde{X}_n(t_I)]\} + \underset{\rightarrow}{\mathrm{Var}}\{\mathrm{Im}[\tilde{X}_n(t_I)]\} \qquad (38)$$

the asymptotic mean value and variance can be experimentally deduced through eqns.3, 8, 15. By supposing, for example, a real signal with only one spectral component ($x(t) = \mathrm{e}^{\mathrm{j}2\pi f_1 t} + \mathrm{e}^{-\mathrm{j}2\pi f_1 t}$), the asymptotic real and imaginary part of the estimator of the spectral component $\mathrm{e}^{\mathrm{j}2\pi f_1 t}$, and the shape of the asymptotic variance for a rectangular window due to the other spectral component $\mathrm{e}^{-\mathrm{j}2\pi f_1 t}$:

$$\sigma^2 = |X_1|^2 |W(2f_1T_s)|^2 \qquad (39)$$

can be deduced as a function of fT_s by considering different values of $2f_1T_s$.

4. CONCLUSIONS

The errors due to the sampling strategy and the filtering technique in the digital spectrum analysis depend on the initial sampling instant, which a priori is unknown. According to the Bayesian approach, this initial instant can be assumed as a random variable with an uniform distribution because any prior information on it is missing. In order to avoid the influence of the conventional time origin on the instrument performance, the asymptotic statistical parameters, i.e. the asymptotic mean and the asymptotic mean-square error, of the estimator of each spectral component must be deduced. The method previously introduced for the rectangular window [4] was modified in order to find a more general approach which could be adopted also for the cascade of rectangular windows. The deduced expression of asymptotic variance shows that the contribution of the spectral components different from the estimated one can be deduced by summing the squared magnitude of each spectral component multiplied by a weighting function whose expression is given for the a rectangular window and for the cascade of two or three rectangular windows. Beside the results relative to the resolution, frequency leakage, and sampling theorem are given in order to compare the different proposed windows. The experimental results obtained by simulating the proposed procedure were in very good agreement with the theoretical findings.

5. APPENDIX

From eqn.23 the statistical mean value of the estimator $\widetilde{X}_n$ results:

$$\mathrm{E}\left\{\widetilde{X}_n^{(2)}\right\} = X_n + \mathrm{E}\left\{\Delta_n^{(2)}(t_I)\right\} \quad \text{(A1)}$$

where $E\left\{\Delta_n(t_I)\right\}$ is the bias. On the other hand, from eqn.18 it can be deduced:

$$\mathrm{E}\left\{\Delta_n^{(2)}(t_I)\right\} = \\ = \frac{1}{N_1 N_2}\sum_{\substack{q=-M\\q\neq n}}^{+M} X_q \mathrm{E}\left\{\mathrm{e}^{\mathrm{j}2\pi(q-n)f_1 t_I}\right\}\sum_{p=1}^{N_1}\sum_{k=1}^{N_2}\mathrm{e}^{\mathrm{j}2\pi(q-n)f_1(k+p-1)T_s} \quad \text{(A2)}$$

The continuous random variable t_I, uniformly distributed in the interval $\pm T/2$, has the following characteristic function:

$$\mathrm{E}\left\{\mathrm{e}^{\mathrm{j}2\pi\alpha t_I}\right\} = \int_{-T/2}^{+T/2} f(t)\mathrm{e}^{\mathrm{j}2\pi\alpha t}\mathrm{d}t = \frac{\sin(\pi\alpha T)}{\pi\alpha T} = \mathrm{sinc}(\alpha T) \quad \text{(A3)}$$

where $f(t) = 1/T$ is the uniform probability density. Consequently the limiting value of eqn.A3 for T tending to infinity with $\alpha = (q-n)f_1$ is:

$$\lim_{T\to\infty}\mathrm{E}\left\{\mathrm{e}^{\mathrm{j}(q-n)2\pi f_1 t_I}\right\} = \begin{cases}1 & \text{for } q - n = 0\\ 0 & \text{elsewhere}\end{cases} \quad \text{(A4)}$$

Since in the double sum of eqn.A2 the quantity $q-n$ must be always different from zero, the asymptotic bias results null:

$$\lim_{T\to\infty}\mathrm{E}\left\{\Delta_n(t_I)\right\} = \underset{\to}{\mathrm{E}}\left\{\Delta_n(t_I)\right\} = 0 \quad \text{(A5)}$$

and consequently the asymptotic mean of $\widetilde{X}_n$ becomes:

$$\lim_{T\to\infty}\mathrm{E}\left\{\widetilde{X}_n(t_I)\right\} = X_n \quad \text{(A6)}$$

Because the asymptotic bias is null, the asymptotic mean-square error of $\widetilde{X}_n$ coincides with its asymptotic variance. By recalling eqn.23 and taking into account eqn.A5, it can be deduced:

$$\underset{\to}{\mathrm{Var}}\left\{\widetilde{X}_n\right\} = \underset{\to}{\mathrm{Var}}\left\{\Delta_n(t_I)\right\} = \underset{\to}{\mathrm{E}}\left\{\Delta_n(t_I)\Delta_n^*(t_I)\right\} \quad \text{(A7)}$$

By recalling eqn.24, the asymptotic expected value of $\Delta_n(t_I)\Delta_n^*(t_I)$ is given by:

$$\underset{\to}{\mathrm{E}}\left\{\Delta_n(t_I)\Delta_n^*(t_I)\right\} = \sum_{\substack{q=-M\\q\neq n}}^{+M}\sum_{\substack{q'=-M\\q'\neq n}}^{+M} X_q X_{q'}^* \underset{\to}{\mathrm{E}}\left\{\mathrm{e}^{\mathrm{j}2\pi(q-q')f_1 t_I}\right\}\cdot \\ \cdot W((q-n)f_1T_s)W^*((q'-n)f_1T_s) \quad \text{(A8)}$$

In analogy to eqn.A4 it results:

$$\lim_{T\to\infty}\mathrm{E}\left\{\mathrm{e}^{\mathrm{j}2\pi(q-q')f_1 t_I}\right\} = \begin{cases}1 & \text{for } q = q'\\ 0 & \text{elsewhere}\end{cases} \quad \text{(A9)}$$

Therefore eqn.A8 becomes:

$$\underset{\to}{\mathrm{E}}\left\{\Delta_n(t_I)\Delta_n^*(t_I)\right\} = \sum_{\substack{q=-M\\q\neq n}}^{+M}\left|X_q\right|^2\left|W((q-n)f_1T_s)\right|^2 \quad \text{(A10)}$$

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