COHERENCE COEFFICIENTS AS UNCERTAINTY PARAMETERS OF ERROR VALUE SET

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ABSTRACT

Arithmetical operations performed on measurement results treated as intervals cause that the influence both errors distribution shape and errors correlation to the width of the resultant interval must be considered. The paper describes the use of reductive interval arithmetic [2] in such a situation. Coherence coefficients which generally describe interdependence between intervals are interpreted as parameters of the error value set. The composition rule for two kinds of coefficients – those dependent on the shapes of the error distributions and those dependent on correlation between errors – is presented. Uncertainty calculation inaccuracy of the described method has been analysed in the paper, too.

Keywords: Reductive Interval Arithmetic, Measurement Error, Uncertainty

1. INTRODUCTION

The most common manner of presenting the measurement result is its interpretation as an interval [5], as it is shown in Fig. 1. The midpoint of the interval is the evaluation of the measurand value $\hat{x}$ that is in particular conditions the nearest number to the true value $\dot{x}$ of the measurand x. To calculate the interval bounds, the knowledge of its radius value is necessary. The radius is called uncertainty in this case and denoted by Δ. The uncertainty is defined in such a way, that the probability α of the occurrence of the true value $\dot{x}$ within this interval is properly high.

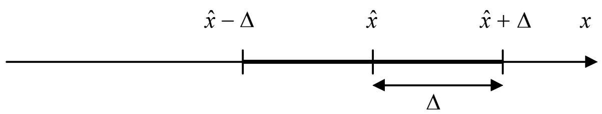


Fig.1. Interval interpretation of the measurement result uncertainty Δ_x, $\hat{x}$ is evaluation of the measurand value

Specified interval parameters are traditionally calculated basing on probability density function describing the measurement results set [5]. However, determining this function is not always possible, for example when measurand changes in time [1, 3]. In the paper [2] it is proposed to define uncertainty Δ not as a parameter of the measurand set, but as a parameter of the set of error values. The error of the evaluation $\hat{x}$ of the measurand is defined as

$$\delta := \hat{x} - \dot{x}, \tag{1}$$

under assumption that it is additive to $\hat{x}$. The measurand evaluation error δ forms a set of values which can be determined in measurement, analytic or simulative way. Assuming that the probability density function $g(\delta)$ of the error value set has non-negative and bounded values and this function is symmetrical in relation to zero and decreasing to zero for $\delta \to \pm\infty$, the uncertainty is defined as

$$\frac{1}{F}\int_{-\Delta}^{+\Delta} g(\delta)d\delta = \alpha, \tag{2}$$

where α is the confidence level called here the uncertainty level. Moreover

$$F = \int_{-\infty}^{+\infty} g(\delta)d\delta = 1. \tag{3}$$

The error δ and the evaluation $\hat{x}$ are additive, therefore the measurement result as an interval can be treated as a sum of two components – the evaluation $\hat{x}$ and the interval $[-\Delta, +\Delta]$, called uncertainty interval. So, the measurement result can be described as

$$x = [\hat{x}-\Delta\ ,\ \hat{x}+\Delta] = \hat{x} + [-\Delta\ ,+\Delta] = \hat{x} \pm \Delta. \tag{4}$$

Additivity enables to perform operations on measurement results separately for evaluations and for uncertainty intervals. It is exceptionally important in the case of considering algorithms as processing sequences of measurement data being instantaneous values of signals varying in time. That is because errors of different kind are propagated from the input to the output in a different way [1, 2]. In such cases the evaluation error can be described as the sum of the partial errors $\delta_1, \delta_2, \ldots, \delta_N$:

$$\delta = \delta_1 + \delta_2 + \ldots + \delta_N. \tag{5}$$

In order to calculate the uncertainty of the final error described by equation (5), two steps are necessary. In the first step the final error distribution is determined. It consists in convoluting the probability density functions of the partial errors, as in expression

$$g(\delta) = g_1(\delta_1) * g_2(\delta_2) * \ldots * g_N(\delta_N), \tag{6}$$

where * is the symbol of convolution. In the second step the final error uncertainty is calculated basing on the definition and the assumed uncertainty level α. For this purpose the expression (2) must be solved which can be generally described as

$$\Delta = U_\alpha\left[g(\delta)\right]. \tag{7}$$

Convolution is in general case a complex task, therefore in [2] such a mathematical method has been proposed that enables substitution of the operations on error distributions with operations directly performed on their uncertainties. It consists in replacing the couple of equations (6) and (7) with one approximate expression which can be written in the general form as

$$\Delta = f_\Delta\left(\Delta_1, \Delta_2, \ldots, \Delta_N\right). \tag{8}$$

Its accuracy should be so high that the approximation errors are nonessential comparing to other factors, which limit the accuracy of uncertainty calculations, for example, an accuracy of error sources identification.

The main advantage of an approximating expression usage is the simplicity of calculations, essentially important in complex processing conditions. One of the methods of approximation is offered by the reductive interval arithmetic [2]. It provides matrix equations convenient in computer implementations.

2. REDUCTIVE INTERVAL ARITHMETIC

The interval is a bounded set of real numbers. In reductive arithmetic interval x is defined by pair of numbers: the midpoint $\breve{x} \equiv \mathrm{mid}(x) \in \boldsymbol{R}$, $\boldsymbol{R}$ is the set of real numbers and the radius $\mathrm{rad}(x) \in \boldsymbol{R}^{0+}$, where $\boldsymbol{R}^{0+}$ is a set of real non-negative numbers. The interval is the set $x \equiv \left[\underline{x}\ ,\ \overline{x}\right]$ whose lower bound is defined as

$$\underline{x} := \breve{x} - \mathrm{rad}(x) \tag{9}$$

and the upper bound

$$\overline{x} := \breve{x} + \mathrm{rad}(x). \tag{10}$$

Therefore the interval can be written as

$$\begin{aligned} x &\equiv \left[\breve{x} - \mathrm{rad}(x), \breve{x} + \mathrm{rad}(x)\right] \equiv \\ &\equiv \breve{x} + \left[-\mathrm{rad}(x), \mathrm{rad}(x)\right] \equiv \breve{x} \pm \mathrm{rad}(x). \end{aligned} \tag{11}$$

Expression (11) describes the interval as a sum of the interval midpoint and so called unloaded interval. The midpoint of the unloaded interval is zero, so this interval is defined only by its radius $\mathrm{rad}(x)$. In the short form unloaded interval is represented as $\pm\mathrm{rad}(x)$.

Arithmetical operations on intervals using reductive interval arithmetic are carried out assuming that all unloaded intervals are interdependent. The dependence between unloaded intervals is described by the coherence coefficient r. So, in the set $\left\{\pm\mathrm{rad}(x_1), \pm\mathrm{rad}(x_2)\right\}$ containing two unloaded intervals they are described adequately by the pairs of numbers $\left(\mathrm{rad}(x_1), r_{12}\right)$ and $\left(\mathrm{rad}(x_2), r_{21}\right)$, additionally it is $r_{12} = r_{21}$. In the set of N unloaded intervals every one is described by the set of N numbers: the radius and $N-1$ coherence coefficients. For example for the interval number 1 this set has a form as follows

$$\left(\mathrm{rad}(x_1), r_{12}, r_{13}, \ldots, r_{1N}\right). \tag{12}$$

Relations between unloaded intervals are of geometrical nature. It is assumed that the set of unloaded intervals is a representation of the equally numerous set of vectors in N-dimensional space, and the vectors have the common origin. They are called radius vectors and denoted as $\vec{\mathrm{rad}}(x_1), \vec{\mathrm{rad}}(x_2), \ldots, \vec{\mathrm{rad}}(x_N)$. For each pair of vectors $\vec{\mathrm{rad}}(x_i)$ and $\vec{\mathrm{rad}}(x_j)$, $i, j = 1, \ldots N, i \neq j$ there is determined the angle γ_{ij} between them such that

$$\gamma_{ij} = \gamma_{ji},\ \pi \geq \gamma_{ij} \geq 0. \tag{13}$$

The applied mapping of the set of the radius vectors on the set of the unloaded intervals is based on the assumption that mathematical dependencies describing relations between interval radii are obtained basing on relations between particular radius vectors. This mapping is described by the following relations

$$\mathrm{rad}(x_i) = \left|\vec{\mathrm{rad}}(x_i)\right|, \tag{14}$$

$$r_{ij} = \cos\gamma_{ij}. \tag{15}$$

It means that all operations on unloaded intervals are preceded by adequate operations on radius vectors and the resultant vector is mapped on the unloaded interval in the meaning of equation (14) and (15). From the relation (13) and (15) it follows that

$$-1 \leq r_{ij} \leq 1 \text{ and } r_{ij} = r_{ji}. \tag{16}$$

The basic operation on unloaded intervals is determination of its sum. Let us take intervals from the set $\left\{\pm\mathrm{rad}(x_1), \pm\mathrm{rad}(x_2), \ldots, \pm\mathrm{rad}(x_N)\right\}$. In the short form the sum of them can be written as

$$\begin{aligned} &\pm\mathrm{rad}(y) = \pm\mathrm{rad}(x_1) + \pm\mathrm{rad}(x_2) + \ldots + \pm\mathrm{rad}(x_N) = \\ &= \pm\mathrm{rad}(x_1) \pm \mathrm{rad}(x_2) \ldots \pm \mathrm{rad}(x_N) \end{aligned} \tag{17}$$

Addition of unloaded intervals corresponds to addition of radius vectors as it is described by the following equation

$$\vec{\mathrm{rad}}(y) = \vec{\mathrm{rad}}(x_1) + \vec{\mathrm{rad}}(x_2) + \ldots + \vec{\mathrm{rad}}(x_N). \tag{18}$$

The radius $\mathrm{rad}(y)$ of the resultant interval is equal to the resultant vector length $\left|\vec{\mathrm{rad}}(y)\right|$. It means that $\mathrm{rad}(y)$ is calculated basing on geometrical dependencies between radius vectors from which one obtains [2]

$$\mathrm{rad}(y) = \sqrt{\mathbf{rad}(x)^{\mathrm{T}}\,\mathbf{R}\,\mathbf{rad}(x)}, \tag{19}$$

where T is the transposition symbol, and moreover

$$\mathbf{rad}(x) = \begin{bmatrix} \mathrm{rad}(x_1) \\ \mathrm{rad}(x_2) \\ \vdots \\ \mathrm{rad}(x_N) \end{bmatrix},\ \mathbf{R} = \begin{bmatrix} 1 & r_{12} & r_{13} & \cdots & r_{1N} \\ r_{21} & 1 & r_{23} & \cdots & r_{2N} \\ & & \vdots & & \\ r_{N1} & r_{N2} & r_{N3} & \cdots & 1 \end{bmatrix}. \tag{20}$$

$\mathbf{R}$ is the coherence matrix consisting of coherence coefficients of component intervals, and $r_{ij} = 1$ for $i = j,\ i, j = 1, \ldots, N$.

For two intervals equation (11) has the following form

$$\mathrm{rad}(y) = \sqrt{\mathrm{rad}^2(x_1) + \mathrm{rad}^2(x_2) + 2\mathrm{rad}(x_1)\mathrm{rad}(x_2)\,r_{12}}. \tag{21}$$

Eq. (21) shows that the resultant interval radius depends on the value of the coherence coefficient. As it is the cosine function (15) of the angle between proper vectors of the partial interval radii one can distinguish three characteristic situations presented in Fig.2. The first one takes place when both radius vectors have the same direction which means they lie on the same straight line and have the same sense (Fig.2a). In this case $\gamma_{12} = 0$, which means that $r_{12} = cos\gamma_{12} = 1$ so, according to Eq. (21), the expression for the resultant radius is of the form

$$\text{rad}(y) = \sqrt{(\text{rad}(x_1) + \text{rad}(x_2))^2} = \text{rad}(x_1) + \text{rad}(x_2) . \quad (22)$$

In this case the resultant interval radius is the sum of the partial interval radii.

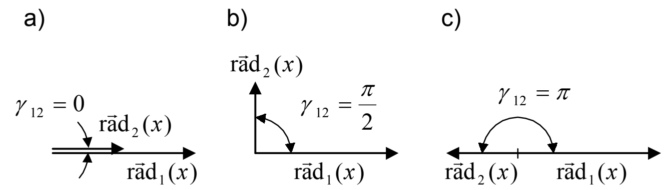


Fig.2. Characteristic cases of mutual orientation of two radius vectors, a) for $r_{12} = 1$*, b)* $r_{12} = 0$*, c)* $r_{12} = -1$

For perpendicular vectors of the partial interval radii (Fig.2b) one obtains $\gamma_{12} = \frac{\pi}{2}, r_{12} = 0$. Eq. (21) in this case has the form

$$\text{rad}(y) = \sqrt{\text{rad}^2(x_1) + \text{rad}^2(x_2)} . \quad (23)$$

In the case when vectors lie on the same straight line (Fig.2c) and have opposite senses, one obtains $\gamma_{12} = \pi, r_{12} = -1$. The resultant interval length is in this case calculated using the following expression

$$\text{rad}(y) = \sqrt{(\text{rad}(x_1) - \text{rad}(x_2))^2} = |\text{rad}(x_1) - \text{rad}(x_2)| \quad (24)$$

which means that the resultant interval radius equals the absolute value of the difference of the component interval radii.

Let us consider the sum two intervals x_1 and x_2

$$y = x_1 + x_2 . \quad (25)$$

According to Eq. (11), all intervals in this equation can be expressed as a sum of the midpoint and the unloaded interval. Thus,

$$\breve{y} \pm \text{rad}(y) = \breve{x}_1 \pm \text{rad}(x_1) + \breve{x}_2 \pm \text{rad}(x_2) . \quad (26)$$

Taking into consideration the independence of the midpoint and the radius, the above relationship can be expressed by two equations. One of them in the form

$$\text{mid}(y) = \breve{y} = \breve{x}_1 + \breve{x}_2 \quad (27)$$

describes the relation between the midpoints, whereas the second one

$$\pm\text{rad}(y) = \pm\text{rad}(x_1) \pm \text{rad}(x_2) \quad (28)$$

determines the interdependence of the radii. The interval midpoints are real numbers and their summing is performed according to the rules of elementary arithmetic. However, the radius of the resultant interval is given by Eq. (21).

Other elementary arithmetical operations on intervals conforming to reductive arithmetic rules are described in [2].

Example 1. Consider the sum of two intervals $x_1 = [-2,-1]$ and $x_2 = [3,4]$. Using the rules of classical interval arithmetic [5] the bounds of the sum of the exemplary intervals are $\underline{y}^{\text{cl}} = -2 + 3 = 1$ and $\overline{y}^{\text{cl}} = -1 + 4 = 3$ (the symbol „cl" in the superscript means classical interval arithmetic). Then, the sum of these intervals is $x^{\text{cl}} = [1,3]$.

When using the reductive arithmetic one should first present the both component intervals as a sum of the midpoint and the unloaded interval. Using the short notation of the exemplary intervals one obtains $x_1 = -1{,}5 \pm 0{,}5$ and $x_2 = 3{,}5 \pm 0{,}5$. According to Eq. (27) the midpoint value of the resultant interval is $\breve{y} = \breve{x}_1 + \breve{x}_2 = -1{,}5 + 3{,}5 = 2$, and its radius value is $\pm\text{rad}(y) = \pm\text{rad}(x_1) \pm \text{rad}(x_2) = \pm 0{,}5 \pm 0{,}5$. Basing on Eq. (28), for the coherence coefficient equal to 1 the value of the resultant interval radius is $\text{rad(y)} = \text{rad}(x_1) + \text{rad}(x_2) = 0{,}5 + 0{,}5 = 1$. If the coefficient equals -1, then according to equation (24) $\text{rad}(y) = |\text{rad}(x_1) - \text{rad}(x_2)| = |0{,}5 - 0{,}5| = 0$. It means that the resultant interval varies from $y = 2 \pm 1 = [1,3]$ to $y = 2$ according to the coherence coefficient value, which changes from 1 to –1. In the first case one obtains the same interval as that calculated by means of classical interval arithmetic. In the other one obtains the real number, that is the interval of the radius equal to zero.

3. DETERMINIG THE UNCERTAINTY USING REDUCTIVE INTERVAL ARITHMETIC

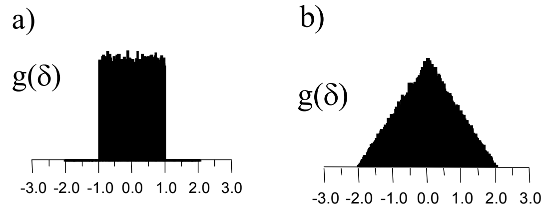
In the reductive arithmetic categories the measurement result is the sum of two components: the midpoint, which is the evaluation $\breve{x}$ of a measurand value and the unloaded interval $\pm\Delta$, with the radius equal to the uncertainty Δ. This unloaded interval is called an uncertainty interval. In such a case the evaluation and the uncertainty can be determined separately. Therefore, the uncertainty of the final error, which is the sum of errors described by expression (5), can be calculated by operations only on partial uncertainties $\Delta_1, \Delta_2, \ldots, \Delta_N$ according to the equation

$$\Delta = \sqrt{\mathbf{u}^{\text{T}} \mathbf{R}\, \mathbf{u}} \quad (29)$$

obtained on the basis of Eq. (19), where $\mathbf{R}$ is the coherence matrix and

$$\mathbf{u} = [\Delta_1 \;\; \Delta_2 \; \ldots \; \Delta_N]^{\text{T}} . \quad (30)$$

In the situation considered the uncertainty is calculated in two steps. First step is to determine partial uncertainties basing on the definition (2). In the second step the final uncertainty is calculated according to the matrix expression (29). In order to perform the mentioned above calculations, the coherence matrix **R** is necessary to be known. It is interpreted as a numerical expression of interdependencies between errors when error sets are summed accordingly with Eq. (5).



Rys.3. Uniform distribution of the error value set a), resultant distribution of the sum of two noncorrelated errors having identical uniform distribution b)

Coherence coefficients of uncertainty intervals are of two kinds. The first kind corresponds with the shape of distribution of the errors represented by these intervals. As an example let us assume that the coherence coefficient describes interdependence of two error value sets during its convolution. The errors have identical rectangular distribution like the one presented in Fig. 3a and described by the expression

$$g_R(\delta) = \begin{cases} \dfrac{1}{h} & \text{for} \quad -\dfrac{h}{2} \le \delta \le \dfrac{h}{2} \\ 0 & \text{for} \quad \dfrac{h}{2} < \delta < -\dfrac{h}{2} \end{cases} \tag{31}$$

As a result of convolution one obtains [2] the distribution

$$g_T(\delta) = \begin{cases} 0 & \text{for} \quad h \le \delta \le -h \\ \dfrac{h+\delta}{h^2} & \text{for} \quad -h \le \delta \le 0 \\ \dfrac{h-\delta}{h^2} & \text{for} \quad 0 \le \delta \le h \end{cases} \tag{32}$$

presented in Fig.2b. Basing on it, one can calculate the uncertainty using Eq. (2). For the error having rectangular distribution (31) one finds

$$\alpha = \int_{-\Delta_R}^{\Delta_R} \frac{1}{h} d\delta . \tag{33}$$

After transformations, assuming that $\alpha = 0{,}95$, the expression describing the uncertainty Δ_R of the rectangular distribution is

$$\Delta_R = \alpha \frac{h}{2} = 0{,}95 \frac{h}{2} = 0{,}475h . \tag{34}$$

When using definition (2) to calculate uncertainty Δ_T of the error having triangle distribution (32), one obtains [2]

$$\Delta_T = 0{,}776h . \tag{35}$$

The triangle distribution (32) is a convolution of two identical rectangular distributions, so it describes the resultant set of error values which is the composition of two sets with uniform distribution. When the final and partial uncertainties are known, one can calculate coherence coefficient. Basing on equation (29), for $N = 2$ there is

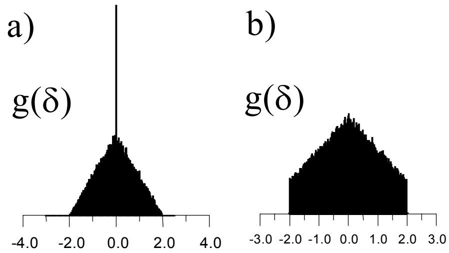
$$\Delta = \sqrt{\Delta_1^2 + \Delta_2^2 + 2\Delta_1\Delta_2 r_{12}} . \tag{36}$$

In the considered situation the partial uncertainties are the same and equal to the rectangular distribution uncertainty, so there is $\Delta_1 = \Delta_2 = \Delta_R$. The final uncertainty has the triangle distribution, therefore $\Delta = \Delta_T$. Introducing these dependencies to equation (36) and taking into account the uncertainty values from expression (34) and (35), one gets the coherence coefficient equal to

$$r_{12} = \frac{\Delta_T^2}{2\Delta_R^2} - 1 = \frac{(0{,}776h)^2}{2 \cdot (0{,}475h)^2} - 1 = 0{,}334 . \tag{37}$$

Therefore, according to the expression (20), the coherence matrix for two noncorrelated uniform distributions has the form as follows

$$\mathbf{R}_{2R} = \begin{bmatrix} 1 & 0{,}334 \\ 0{,}334 & 1 \end{bmatrix} . \tag{38}$$



Rys.4. Resultant distribution of the sum of two errors having the same rectangular distributions with the correlation coefficient value equal to 0,5 a), equal to −0,5 b)

When correlated sets of error values are composed, the resultant set distribution depends on the correlation coefficient, as it is presented in Fig.3 for two sets of rectangular distribution. The correlation of these sets is obtained in such a way, that the two composed error sets are of identical rectangular distribution and they have a common part denoted as *a*, the values of which are identical for the both sets. Additionally

$$a = 1 - b , \tag{39}$$

where *b* is the residual part of the both sets, the values of which are independent. Considering Eq. (39), one can write $a = r_{cor}$, where r_{cor} is the correlation coefficient calculated in general on the basis of Eq. (42). When partial errors of rectangular distributions are correlated one can obtain description of the resultant distribution basing on simple geometrical relations. For positive correlation there is

$$g(\delta) = \begin{cases} 0 & \text{for} \quad h \le \delta \le -h \\ \dfrac{1-r_{cor}}{h^2}\delta + \dfrac{2-r_{cor}}{2h} & \text{for} \quad -h \le \delta \le 0 \\ -\dfrac{1-r_{cor}}{h^2}\delta + \dfrac{2-r_{cor}}{2h} & \text{for} \quad 0 \le \delta \le h \end{cases} \tag{40}$$

and for the negative one

$$g(\delta)=\begin{cases} 0 & \text{for} \quad h \le \delta \le -h \\ \frac{1-|r_{cor}|}{h^2}\delta+\frac{1-|r_{cor}|}{h} & \text{for} \quad -h \le \delta < 0 \\ |r_{cor}| & \text{for} \quad \delta = 0 \\ -\frac{1-|r_{cor}|}{h^2}\delta+\frac{1-|r_{cor}|}{h} & \text{for} \quad 0 < \delta \le h \end{cases} \qquad (41)$$

Let us consider two sets of errors signed i and j. The correlation coefficient of the errors in these two sets is calculated as

$$r_{ij\text{cor}} = K\frac{\sum_{k=1}^{K}\delta_{ik}\delta_{jk} - \sum_{k=1}^{K}\delta_{ik} - \sum_{k=1}^{K}\delta_{jk}}{\sqrt{(K\sum_{k=1}^{K}\delta_{ik}{}^2-(\sum_{k-1}^{K}\delta_{ik})^2)(K\sum_{k=1}^{K}\delta_{jk}{}^2-(\sum_{k-1}^{K}\delta_{jk})^2)}} \qquad (42)$$

where δ_i, δ_j are values of errors from the two sets numbered as i and j respectively, K is the total number of errors, the same for both sets.

Correlation coefficients form the matrix $\mathbf{R}_{cor}$, where $r_{ij\text{cor}} = r_{ji\text{cor}}$ and $r_{ij\text{cor}} = 1$ for $i = j$ (for $i = j$ the coefficient determines correlation of error with itself). This matrix can be interpreted in reductive interval arithmetic categories as such a coherence matrix which describes only the correlation of the error value sets. Thus, two coherence matrixes can describe the interdependence of the errors. The first one, which can be called the shape matrix, describes properties of noncorrelated errors while they are being composed. The second one, the correlation matrix, describes only correlation of the errors. The uncertainty describing the resultant error depends on both of these matrices. Thus, there is a question how to compose the partial matrices to get the resultant one, which enables to calculate the resultant uncertainty using Eq. (11).

Let us assume that two N dimensional matrices are composed, where $\mathbf{R}_{sh}$ is the shape matrix, and $\mathbf{R}_{cor}$ is the correlation matrix. The coefficients in these matrices are approximately $r_{ij\text{sh}}, r_{ij\text{cor}}$, $i, j = 1, \ldots, N$. The coherence coefficients map angular relationships between the adequate radius vectors, so the composition rule for these coefficients should come from this kind of relations. Let us consider the simplest rule, according to which two angles with indexes i, j (these indexes are omitted for simplicity) are added using an expression

$$\gamma_{res} = \gamma_{sh} + \gamma_{cor} + \gamma_0, \qquad (43)$$

where γ_{res} is the resultant angle and γ_0 is an angle dependent on the initial conditions of the composition procedure. Taking into account that the condition (13) must be complied, the angle between radius vectors is defined as

$$\begin{aligned} \gamma &= \gamma_{res} && \text{for } 0 \le \gamma_{res} \le \pi, \\ \gamma &= 2\pi - \gamma_{res} && \text{for } \pi < \gamma_{res} \le 2\pi. \end{aligned} \qquad (44)$$

The coefficients on the main diagonal of the coherence matrix have always values equal 1, because they define the dependence of the unloaded interval from itself. It means that in this case $\gamma_{sh} = \gamma_{cor} = 0$ and $\gamma_0 = 0$, thus $\gamma = 0$. So, there is $r_{ij} = 1$ for $i = j$. On the other hand, for coefficients laying beyond the main diagonal there is $\gamma_0 = \frac{\pi}{2}$. It comes from the fact, that the coherence coefficient for distributions not changing the shape during its convolution (for example normal distribution) is, as presented in [2], equal to zero. Also for noncorrelated errors this coefficient is equal to zero. Thus, the resultant coefficient is also equal to zero, which imposes the mentioned value of γ_0. Therefore, according to equation (43)

$$\begin{aligned} \cos\gamma_{res} &= \cos(\gamma_{sh}+\gamma_{cor}+\frac{\pi}{2}) = \\ &= \cos\gamma_{sh}\sqrt{1-\cos^2\gamma_{cor}} + \cos\gamma_{cor}\sqrt{1-\cos^2\gamma_{sh}} \end{aligned} \qquad (45)$$

Considering that $r_{sh} = \cos\gamma_{sh}$ and $r_{cor} = \cos\gamma_{cor}$, one obtains the composition rule for the shape and correlation matrices into the resultant coherence matrix as

$$\begin{aligned} r_{ij} &= r_{ij\,\text{sh}}\sqrt{1-r_{ij\,\text{cor}}^2} + r_{ij\,\text{cor}}\sqrt{1-r_{ij\text{sh}}^2} && \text{for } i \ne j, \\ r_{ij} &= 1 && \text{for } i = j. \end{aligned} \qquad (46)$$

The correlation coefficient changes within the limits from -1 to $+1$, and the shape coefficient for typical distributions changes within the limits from 0 (for normal distribution) to 0,334 for two rectangular distributions. For the rectangular and sinusoidal distribution this coefficient is 0,554 [2], but in this case there is an open question what are the terminal values for the correlation coefficient for two different distributions. The next problem is connected with accuracy of Eq. (41). In order to evaluate it, the coherence coefficients of two correlated rectangular distributions obtained in two ways are compared – conforming to the definition (2) and basing on expression (46). The results are presented in Fig.5.

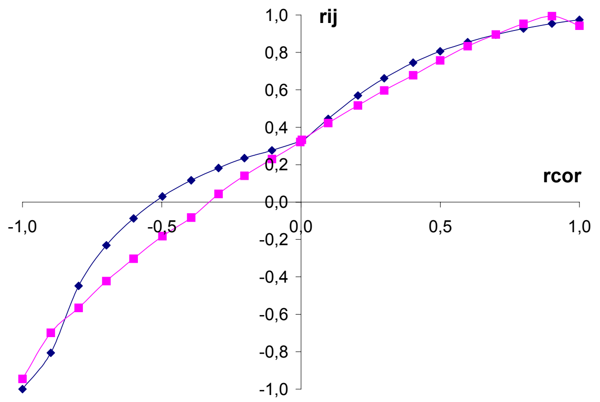


Fig.5. The resultant coherence coefficient as a function of correlation coefficient for two uniform distributions, ♦ calculated applying the definition (2) to Eqs. (40) and (41), ■ calculated when using expression (46)

The graphs in Fig.5 show that the coefficients calculated by using Eq. (46) are not accurate. In order to determine this inaccuracy, the uncertainty calculated basing on the definition (2) was compared with that obtained by using Eq. (36). The resultant coherence matrix was calculated on the basis of expression (46), where the shape matrix is of the form (38). The results are presented in Table 1.

Table 1. *Resultant uncertainty of two errors of rectangular distribution:* Δ_{ideal} *- calculated basing on definition (2) and* Δ *- calculated by using equation (36), where the resultant coherence matrix is calculated basing on the expression (46),* r_{cor} *is the correlation coefficient*

r_{cor}	Δ_{ideal}	Δ	Δ_{ideal}-Δ
1,0	1,90	1,87	0,027
0,9	1,89	1,90	-0,008
0,8	1,88	1,88	-0,001
0,7	1,86	1,85	0,011
0,6	1,84	1,82	0,022
0,5	1,82	1,78	0,036
0,4	1,79	1,74	0,045
0,3	1,74	1,70	0,045
0,2	1,69	1,65	0,039
0,1	1,63	1,60	0,023
0,0	1,56	1,55	0,004
-0,0	1,55	1,54	0,008
-0,1	1,53	1,49	0,038
-0,2	1,50	1,43	0,064
-0,3	1,47	1,37	0,095
-0,4	1,43	1,29	0,139
-0,5	1,37	1,22	0,154
-0,6	1,29	1,12	0,170
-0,7	1,19	1,02	0,164
-0,8	1,00	0,89	0,119
-0,9	0,60	0,74	-0,142
-1,0	0,00	0,31	-0,314

Table 2 presents the comparison of the resultant uncertainty for three error sets of identical rectangular distribution (the shape coefficient in such a case is $r_{sh} = 0{,}192$ [2]) for selected values of the correlation coefficients.

Table 2. *Resultant uncertainty for three errors with rectangular distribution:* Δ_{ideal} *- calculated basing on definition (2) and* Δ *- calculated by using Eq. (29) for resultant coherence coefficient obtained basing on Eq. (46),* r_{cor12}*,* r_{cor23}*,* r_{cor13} *are the correlation coefficients calculated by using expression (42)*

r_{cor12}	r_{cor23}	r_{cor13}	Δ_{ideal}	Δ	Δ_{ideal}-Δ
1,0	1,0	1,0	2,85	2,84	-0,011
1,0	0,0	0,0	2,37	2,28	-0,088
0,5	0,5	0,5	2,71	2,51	-0,202
0,0	0,0	0,0	1,95	1,94	-0,009
-0,5	0,5	-0,5	1,67	1,65	-0,017
-1,0	0,0	0,0	0,96	1,29	0,329
-1,0	1,0	-1,0	0,96	0,98	0,025

4. FINAL REMARKS

Simplicity of calculations is the main argument for introducing reductive interval arithmetic to uncertainty calculus. The matrix form of equation (29) enables a clear notation of the described parameters of composed uncertainties. It is exceptionally important in complex measuring conditions, where measurement results are loaded by many different errors, because such a form of equation facilitates calculations by use of a computer. Thus, properties of the presented method enable building uncertainty models of the measuring systems.

Analysis carried out shows that the considered procedure of determining the resultant coherence matrix is not accurate. It results from the fact that the equation (29) approximates the relations between uncertainties while they are being combined. When there is no correlation of the errors, this inaccuracy is, for two rectangular distributions, of the order 1,5% [2]. In the situation when correlation takes place, the inaccuracy is bigger which is illustrated by the presented results. The question is still open if this inaccuracy is excessive. To answer it one should know the inaccuracy of determining the distributions of error sources in measurement praxis. At the moment this kind of research is carried out for typical analog-to-digital processing chains.

If the accuracy of the procedure (46) is not sufficient, one can search better relations to approximate combining coherence coefficients. The curves presented in Fig.4 illustrate that using separate relations for positive and negative values of correlation coefficient could enable more accurate approximation. The research in this topic is to be continued.

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