Fuzzy Approach for the Theory of Measurement Inexactness

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ABSTRACT

In this paper we propose the weak t-norm (wt-norm) based arithmetics to describe the propagation of systematic as well as statistical inexactness. We use the general term "inexactness" to express all possible approaches of inaccuracy evaluations. In the presented theory the measurement results are fuzzy intervals. The arithmetic operators are obtained by applying the generalisation of Zadeh's extension principle based on the wt-norm. The averaging operator defined in the framework of wt-norm arithmetic decreases the statistical component, and does not change the systematic component of inexactness. We are of the opinion that statistical method, preferred in the Guide, is not adequate to describe the systematic effects.

Key words: fuzzy set theory, measurement inexactness, theory of measurement, non-statistical uncertainty estimation, measurement uncertainty, wt-norm, t-norm.

1. INTRODUCTION

The measurement is an estimation based on inexact[1] operations depending on the model of measured quantities [1,2].

In the framework of standard approach given in the Guide [3], the result of measurement is a random variable, the inexactness is described using the statistical methods, and uncertainty is the means of inexactness estimation. However the systematic component of error is not a random variable, so another non-statistical model is necessary.

In the error theory, the systematic error is described using the interval arithmetics for limiting error propagation.

[1] Uncertainty or limiting error is a method of describing the measure of inexactness.

Since Zadeh introduced the fuzzy set theory we have been able to find some papers which present applications of fuzzy approaches to inexactness of measurements (errors, uncertainty [4,5,6,7]). Fuzzy set theory with fuzzy arithmetic (which appears as a generalisation of interval arithmetics) is a good model for description of systematic errors, but is not appropriate for random component [6]. The fuzzy arithmetics has the same weakness as the interval arithmetics: the support of sum of fuzzy intervals is the sum of supports: the fuzzy arithmetics with Zadeh extension principle describes properly only propagation of systematic error.

Comprehensive description of statistical and systematic component is possible by the use of random fuzzy variables. But contrary to the classical statistical methods, no unique statistical theory of fuzzy random variables, has been established as yet. So far we have not found in literature any complete theory of both, random and systematic component of inexactness. Therefore we propose the weak t-norm (wt-norm) based arithmetic to describe the propagation of systematic as well as statistical inexactness.

2. MATHEMATICAL MODEL OF MEASUREMENT

The measurement is a mapping $\boldsymbol{f}$ from the space of events (objects, phenomena) Ω into the mathematical space $\boldsymbol{Y}$ which represents the results of measurement (values of physical quantities):

$\boldsymbol{f}: \Omega \rightarrow \boldsymbol{Y}$

Mathematical qualities of measurement model depend on the assumption referred to the qualities of event space Ω, measured results $\boldsymbol{Y}$, and measuring function $\boldsymbol{f}$. It rises the following questions:

1) What is the nature of event space Ω ?
2) Is the measuring function $\boldsymbol{f}$ deterministic or random?
3) Is the space of measured results $\boldsymbol{Y}$ the set of real numbers or fuzzy intervals?

It is necessary to answer these questions in order to get familiar with the model of measuring the inexactness

3. MODELS OF INEXACTNESS

The process of evaluating of measure of inexactness consists of three steps:

1) Estimation of parameters of single measurement inexactness model. This process has an expert character, and is based on the model of measuring instrument. The single measurement is a single comparison process.
2) Evaluation of propagation of inexactness because of mathematical operations performed on single "pure" result. Usually we do not use pure results, and the final result of measurement is obtained in some mathematical operations (averaging, filtering, etc.)
3) Estimation of confidence interval width (expanded uncertainty or fuzzy uncertainty) for given degree of confidence (probability or possibility level).
In the existing inexactness theories we can find the following assumptions of models of measurement:

1) statistical model: the measuring function ***f*** is a random function, but the measured results are real numbers (crips numbers). This is the standard statistical model of uncertainty measurement. The systematic component of inexactness is not correctly represented in this model.

2) fuzzy set model: the measuring function is fuzzy, and the space ***Y*** of measured results is the fuzzy intervals. In this model:

a) results of single measurement are fuzzy intervals; the exactness is described by membership function,

b) mathematical operations, which we make on the measuring results, are operations perfomed on fuzzy intervals. In order to define arithmetic of fuzzy intervals we use the extension principle. In the theory of fuzzy sets one defines the extension principle based on t-norms.

c) random-fuzzy model: the measuring function ***f*** is a random function, and the space ***Y*** of measured results is the set of fuzzy intervals ***F***. In this models measurement results are random fuzzy intervals. In this model we can define the extended uncertainty for a given significance level and degree of possibility. The random component of inexactness is described by random properties of function ***f*** (described by the function of probability distribution). The systematic component is given by the possibility function, which is determined by experts with respect to the model of measuring device.

In this paper we consider only fuzzy model of inexactness. The main problem of propagation of inexactness is the behaviour of systematic and random component of inexactness. In this model it depends on the arithmetic of fuzzy intervals; we can find three possibilities:

a) extension principle based only on ***min*** operator (classical Zadeh's extension principle) for the model of measurement uncertainty. This model is presented in [1], but it is correct only for systematic errors.
b) extension principle based on t-norms (the ***min*** operator is the strongest t-norm). In paper [2] we propose such model and we show that such fuzzy arithmetic describe properly propagation as well as the systematic and random component of inexactness.
c) now in this paper we propose the weak t-norm (which is an extension of the t-norm) in order to describe arbitrary contraction of spread of fuzzy intervals (in the case of strong correlation of data).

4. ARITHMETIC OF FUZZY INTERVALS

4.1. Definitions

In this paper we consider fuzzy sets on $\Re$. We represent the fuzzy sets using capitals, e. g. ***A, B, C***….

We denote by $[A]^\alpha = \{x\in\Re: A(x) \geq \alpha\}$ the α-cut of ***A***, where $\alpha\in(0,1]$ and $A(\cdot)$ is membership function of ***A***. We separately define 0-cut, written $[A]^0$, as the support of ***A***:

supp$A$ = cl($\{x\in\Re: A(x) > 0\}$).

The 1-cut defines the kernel of ***A***: ker $A = [A]^1$.

Among fuzzy sets we distinguish the class ***FI*** of fuzzy intervals. $A\in$ ***FI*** if $[A]^1 \neq \emptyset$ and $[A]^\alpha$ is closed bounded interval for $\alpha\in[0,1]$.

Let's consider a fuzzy interval ***A*** with kernel $[A]^1 = [a_K, a_M]$ and the support $[A]^0 = [a_L, a_R]$. Of course $a_L \leq a_K \leq a_M \leq a_R$; $\beta_L = a_K - a_L$ and $\beta_R = a_R - a_M$ are called the left and right spread of ***A***. When there is only one real number $m = m(A)$ such that $A(m) = 1$, ***A*** is called a fuzzy number.

A special type of fuzzy intervals is obtained when $A(x)$ is linear on $[a_L, a_K]$ and $[a_M, a_R]$. We shall call them trapezoidal fuzzy intervals, and use the notation

$A = (a_L, a_R, \beta_L, \beta_R)$.

Notice that real numbers can be seen as cases of sets with zero spreads.

A weak triangular norm (shortly, wt–norm) T is a mapping $T: [0,1]^2 \to [0,1]$ with the following properties:

(i) T is symmetric, associative, non-decreasing in both components,

(ii) $(\forall x\in[0,1))\ T(x,1) \leq x$, $T(1,1) = 1$.

Let's remark that t-norms are special types of wt-norms. Namely, it holds that

(iii) $(\forall x\in[0,1])\ T(x,1) = x$.

The greatest t-norm $T_M(x,y) = \min(x,y)$ is the greatest wt-norm. The smallest wt-norm T_w is defined by:

$$T_w(x,y)=\begin{cases}1 & \textit{if} \quad x=y=1\\ 0 & \text{elsewhere}\end{cases}$$

Therefore, for each wt-norm T we always have

$T_W(x,y) \le T(x.y) \le T_M(x,y)$.

Now we are interested in a case when the wt-norm T is not a t-norm. Such mapping satisfies the condition:

(iv) $(\exists x\in(0,1))\ T(x,1) < x$.

Remark: Let $a\in(0,1)$. Consider a wt-norm T such that $u = T(a,1) < a$. Then

$(\forall x\in[u,a])\ T(x,1) = u$.

Above condition implies that $T(x,1)$ is a discontinuous function of x on (0,1].

4.2. Fuzzy arithmetic based on wt-norms

The definition of arithmetic operations of fuzzy intervals is obtained by applying the generalised version of Zadeh's extension principle.

A real operation $*$: $\Re^2 \to \Re$ is extended to fuzzy intervals by:

$$A * B(z) = \sup_{x*y=z} T\big(A(x),B(y)\big), \quad z\in\Re,$$

where $\boldsymbol{A}$, $\boldsymbol{B}$ are fuzzy sets, and T is a wt-norm.

Let us consider N fuzzy intervals A_i (i=1,…,N) with left β^i_L and right β^i_R spread. For extended addition $\oplus$ T-sum $A = A_1\oplus\ldots\oplus A_N$ based on t-norm T we have:

1) $[A]^1 = [A_1]^1 + \ldots + [A_N]^1$,
2) $[A]^\alpha \subseteq [A_1]^\alpha + \ldots + [A_N]^\alpha$ for $\alpha \in [0,1)$,
3) $\beta_L \ge \max(\beta^1_L, \ldots, \beta^N_L)$, $\beta_R \ge \max(\beta^1_R, \ldots, \beta^N_R)$, for left and right spread of $\boldsymbol{A}$.

Right sides in 1) and 2) are determine by interval arithmetic [8].

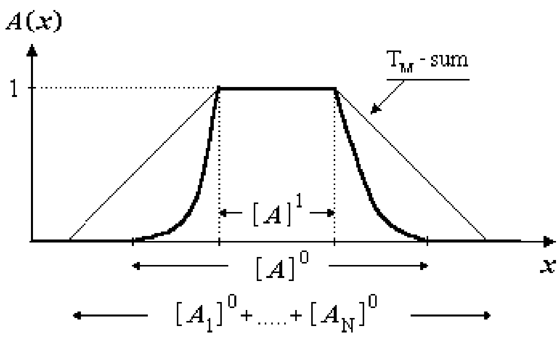

Fig 1. Result of addition of N trapezoidal fuzzy intervals. For T_M norm the sum is trapezoidal, for other t-norms the sum does not have to be trapezoidal (heavy line).

Note that for the addition based on the minimum operator, the resulting spreads are the sum of the incoming spreads (greatest spreads), while for the addition based on the weakest t-norm the resulting spreads are the greatest of the incoming spreads (smallest spreads). For more detailed information on t-norm addition we refer, e.g. to [9]. Stronger contraction of spreads T-sum $\boldsymbol{A}$ can be obtained using wt-norm T. For the smallest wt-norm T_w we obtain T-sum with zero spreads.

5. FUZZY MODEL OF INEXACTNESS OF SINGLE MEASUREMENT RESULTS

In our model of measurement, the single measurement result A_i is a fuzzy interval. Inexactness is described in the term of a membership function $A_i(x)$. The membership function of single measurement is estimated basing on expert knowledge. If we have a series of measuring data obtained in repeatability condition:

$(x_1, x_2, \ldots, x_N)$.

We propose the following procedure to determine the membership function of single measurement result:

1. The support of fuzzy interval A_i characterising the i-th result of measurement is interval of all possible values:

$\mathrm{supp}(A_i) = [x_{min} - \delta, x_{max} + \delta]$,

where: $x_{min} = \min(x_1, x_2, \ldots, x_N)$,

$x_{max} = \max(x_1, x_2, \ldots, x_N)$,

δ - systematic limiting error.

2. Kernel of fuzzy interval A_i is an interval characterising the systematic error:

$\ker(A_i) = [x_i - \delta, x_i + \delta]$.

3. The shape of left and right membership function $A_i(x)$ of fuzzy interval A_i is linear (trapezoid interval).

4. All arithmetical operation on measuring series is made on fuzzy intervals series:

$\{A_i\}^N = (A_1, A_2, \ldots A_N)$.

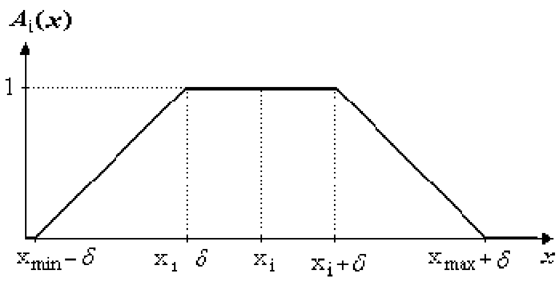

Fig. 2. The membership function of i-th results of measurement.

6. PROPAGATION OF INEXACTNESS COMPONENTS

In order to compute the result of some mathematical operation on measured results we must define the arithmetics of measured results. The main operation is averaging operation E for given series:

$\{x_i\}^N = (x_1, x_2, \dots, x_N)$,

$$x_{AV} = E\left(\{x_i\}^N\right) = \frac{1}{N}\sum_{i=1}^{N} x_i \, .$$

For the probabilistic model, single measurement results x_i are random variables. The arithmetics of random measured results is given by the construction of density distribution function for average estimator. For standard uncertainty case we have a principle:

$$\sigma^2_{X+Y} = \sigma^2_X + \sigma^2_Y + 2\boldsymbol{Cov}(\boldsymbol{X},\boldsymbol{Y}).$$

The measure of uncertainty of x_{av} is the standard deviation $\sigma(x_{va})$.

For fuzzy models, x_i must be replaced in the above equation by fuzzy interval $\boldsymbol{A}_i$:

$$\boldsymbol{E}\left(\{\boldsymbol{A}_i\}^N\right) = \frac{1}{N} \otimes \left(\boldsymbol{A}_1 \oplus \dots \oplus \boldsymbol{A}_N\right),$$

where extended addition $\oplus$ and multiplication $\otimes$ are based on wt-norm.

Making an arithmetical operation on measuring fuzzy intervals we automatically make operation on measures of inexactness.

7. PROPERTIES OF wt-NORM FUZZY MODEL OF INEXACTNESS

In this model the measuring quantities are fuzzy intervals, and the inexactness propagation is described by the fuzzy arithmetics based on the wt-norms. This model gives the unifying theory of two components of inexactness: the systematic and random. We assume that:

1) The measurement of inexactness is described by fuzzy intervals, parameters of its membership function (α-cut) characterise the value of uncertainty.

2) The division into systematic and random component of inexactness is a fundamental assumption.

a) The operation of statistical averaging decreases the random inexactness but does not reduce the systematic one. The measurement result with systematic inexactness component is invariant due to averaging operation. The classes of fuzzy intervals invariant due to averaging we designate as ***FIS*** (systematic fuzzy intervals).

b) A random inexactness can be decreased while averaging all measurement results. In the limit of infinite averaging, all the sequences of fuzzy interval ***FI*** approach a limit to crips fuzzy intervals belonging to ***FIS***.

We define two classes of fuzzy intervals: ***FIS*** – fuzzy systematic intervals and ***FIR*** – fuzzy random intervals. The elements of ***FIS*** are the fuzzy sets $\boldsymbol{S}$ such that $\boldsymbol{S}(x) = 1$ for all $x \in \mathrm{supp}(\boldsymbol{S})$ (crisp sets). The elements of ***FIR*** are the fuzzy sets $\boldsymbol{R}$ such that $\boldsymbol{R}(x)=1$ for only one x. It is possible to prove that the operation of averaging of fuzzy intervals is converging to the crisp set (the law of large numbers for fuzzy variables). This property we interpret as the law of large numbers for statistical component of measuring inexactness (mathematical details one can find in [2]).

Remark. Let $\boldsymbol{S}_1$ and $\boldsymbol{S}_2 \in$ ***FIS*** and $\boldsymbol{R}_1$, $\boldsymbol{R}_2 \in$ ***FIR***. For additions based on wt-norms we have:

1) $[\boldsymbol{S}_1 \oplus \boldsymbol{S}_2]^\alpha = [\boldsymbol{S}_1]^1 + [\boldsymbol{S}_2]^1$ for $\alpha \in [0,1]$,

2) $m(\boldsymbol{R}_1 \oplus \boldsymbol{R}_2) = m(\boldsymbol{R}_1) + m(\boldsymbol{R}_2)$,

3) $[\boldsymbol{S}_1 \oplus \boldsymbol{R}_1]^1 = [\boldsymbol{S}_1]^1 + m(\boldsymbol{R}_1)$.

Low of large numbers for fuzzy sets was studied by several authors, and can be understood in different ways [10].

For given sequence of fuzzy intervals $\{\boldsymbol{A}_i\}^N$ and wt-norm T, we have infinite sequence of average values

$\boldsymbol{E}(\{\boldsymbol{A}_i\}^1)$, $\boldsymbol{E}(\{\boldsymbol{A}_i\}^2)$,...

We shall say that the sequence $\{\boldsymbol{A}_i\}^N$ obey the law of large numbers for wt-norm T if there exist $\boldsymbol{A}_s \in$ ***FIS*** such that:

$$\lim_{N\to\infty} \boldsymbol{E}\left(\{\boldsymbol{A}_i\}^N\right)(\boldsymbol{x}) = \boldsymbol{A}_s(\boldsymbol{x}) \quad \text{for} \quad \boldsymbol{x} \in \Re \, .$$

For the greatest t-norm ***min*** and for sequence of identical fuzzy intervals $\{\boldsymbol{A}\}^N$ there exists no limiting set $\boldsymbol{A_S}$ in this sence, because:

$[\boldsymbol{E}(\{\boldsymbol{A}\}^N)]^\alpha = [\boldsymbol{A}]^\alpha$ for $\alpha \in [0,1]$.

This means that the process of averaging in t-norm ***min*** does not change the inexactness.

The strictly probabilistic model always gives the effect of decreasing the dispersion during the averaging procedure (even in the case of rectangular probability distribution), and for this reason it is not suitable for coherent description of both kinds of inexactness. However either in the classical Zadeh's arithmetic or interval arithmetics all numbers work in the same way in the process of averaging them. The use of the arithmetics based on the t-norms allows to gain the model in which the classes of systematic and random numbers represent different qualities in averaging.

8. NUMERICAL COMPARISON OF STANDARD AND FUZZY METHOD OF COMPUTING OF INEXACTNESS

In order to study the numerical applications of our theory we consider series of N data obtained in a measurement with the use of an instrument characterised by systematic components of inexactness. Lets us assume that these data are characterised by Gauss distribution of probability with mean value x_0 and standard deviation σ. The limiting value of systematic error is δ. We generate such data using random function generator. In table 1 we present the comparison of two measures of inexactness:

the extended uncertainty for probabilistic model, and interval of possibility for fuzzy model.

For extended uncertainty we use the following formula:

$$u_\alpha(x_{AV}) = K_{1-\alpha}\sqrt{s^2(\{x_{AV}\})+\frac{1}{3}\delta^2}\ ,$$

where $s(\{x_{AV}\})$ is the estimator of standard deviation of average of measured series $\{x_i\}$ of data, and K_p – coverage factor for level of confidence p.

The computation of membership function of average value was made using t-norm of Yager T^Y_p

$$T_p^Y(x,y) = 1 - min\left(1, \sqrt[p]{(1-x)^p + (1-y)^p}\right)$$

with parameter p >1 given in table.

For fuzzy trapezoid intervals $\boldsymbol{A}_i=(a^i_L, a^i_R, \beta^i_L, \beta^i_R)$, we have [11]:

$$A_{AV} = E\left(\{A_i\}^N\right) = \left(a_L^{AV}, a_R^{AV}, \beta_L^{AV}, \beta_R^{AV}\right),$$

where

$$\boldsymbol{a}_\gamma^{AV} = \frac{1}{\boldsymbol{N}}\sum_{i=1}^{N}\boldsymbol{a}_\gamma^i \quad \text{and} \quad \beta_\gamma^{AV} = \frac{1}{\boldsymbol{N}}\sqrt[q]{\sum_{i=1}^{N}\left(\beta_\gamma^i\right)^q}\ ,$$

where γ = L,R and $q = \dfrac{p}{p-1} > 1$.

The length of fuzzy confidence interval for possibility level α (half with of α-cut) can be computed from:

$$\mu_\alpha(\boldsymbol{A}_{AV}) = \frac{1}{2}\left[\left(\boldsymbol{a}_R^{AV} - \boldsymbol{a}_L^{AV}\right) + (1-\alpha)\left(\beta_L^{AV} + \beta_R^{AV}\right)\right].$$

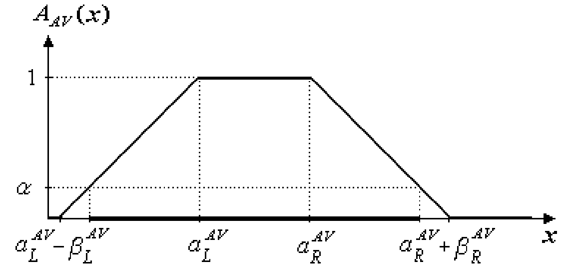

Fig.3. Fuzzy confidence interval (heavy line).

The results of numerical calculations are presented in table 1.

The results of computation of measure of these two methods are very similar for p ∈(2,3), but not identical. Taking into consideration the fact that estimators of variances have statistical dispersion dependent on number of data, we can state that the results obtained in both methods are very similar. But our fuzzy model describes all components of inexactness in a consequent manner.

Table 1. Numerical experiment results.

N	p	α	μ_α	$\boldsymbol{u}_\alpha$
10	1,5	0.01	0.053	0.109
10	2	0.01	0.068	0.109
10	3	0.01	0.098	0.109
100	1,5	0.01	0.022	0.030
100	2	0.01	0.035	0.030
100	3	0.01	0.061	0.030
10	1,5	0.1	0.051	0.059
10	2	0.1	0.065	0.059
10	3	0.1	0.087	0.059

With increasing the number N of averaging data, the width of all α- cuts decreases. The following lemma describes it.

Lemma.

If

1) the following limits exists:

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}a_\gamma^i = a_\gamma^s \quad \text{for } \gamma = L,R,$$

2) and the spreads β^i_L and β^i_R are equi-bounded,

then the limit of sequence of average values exists:

$$\lim_{N\to\infty}\boldsymbol{E}\left(\{\boldsymbol{A}_i\}^N\right)(\boldsymbol{x}) = \boldsymbol{A}_s(\boldsymbol{x}) = \begin{cases} 1 & \text{for } \boldsymbol{x}\in\left[\boldsymbol{a}_L^s, \boldsymbol{a}_R^s\right] \\ 0 & \text{elsewhere} \end{cases}$$

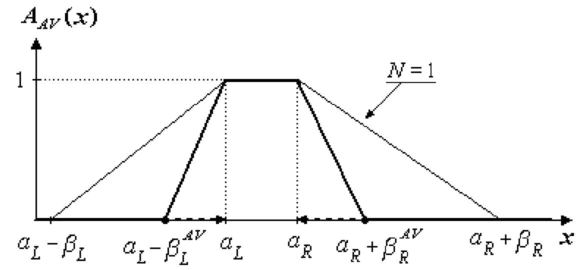

Fig.4. Low of large numbers for identical fuzzy trapezoidal intervals.

Spreads of the average values of identical trapezoidal fuzzy intervals $\mathbf{A}_i = (a_L, a_R, \beta_L, \beta_R)$ are:

$$\beta_\gamma^{AV} = \frac{1}{\sqrt[p]{\boldsymbol{N}}}\beta_\gamma \xrightarrow[N\to\infty]{} 0,$$

for $\gamma = \boldsymbol{L}, \boldsymbol{R}$.

9. CONCLUSIONS

The presented fuzzy theory of measurement inexactness is based on wt-norm arithmetics. For numerical examples we have used the Yager t-norm, but additional studies are necessary for constructing methods of testing t-norm for given empirical data. In order to obtain the correlation effect we introduce weak t-norms; further studies will be performed. In our opinion, wt-norms are necessary to describe a situation when the studied value is obtained as a difference of two measured results, and when we know that the systematic components of inexactness do not change during measurement.

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10. REFERENCES

[1] M. Urbański, "Error as a Parametr of Quantity Model" (in Polish), Metrologia i Systemy Pomiarowe, Vol. 16, Conference Materials, Warszawa, Gliwice 1993, pp. 63-70.

[2] L. Mari, "Notes on the uncertainty of measurement", in: XV IMEKO World Congress, Osaka, Japan, 1999.

[3] Guide to the Expression of Uncertainty in Measurement, ISO, 1993.

[4] G. Mauris, V. Lasserre, L. Foully, "A fuzzy approach for expression of uncertainty in measurement", Measurement 29 (2001), pp. 165-177.

[5] S. Ferson, Lev R. Ginzburg, "Different methods are needed to propagate ignorance and variability", Reliability Engineering and Systems 54 (1996), pp. 133-144.

[6] M. Urbański, J. Wąsowski, "Application of Fuzzy arithmetic to uncertainty measurement theory" (in Polish), Prace Komisji Metrologii Oddziału PAN w Katowicach, seria Konferencje Nr4. Conference Materials, Gliwice-Ustoń 7-9.05.2001, pp.39-50,

[7] X. X. Xia, Z. Wang, Y. Gao, Estimation of non-statistical uncertainty using fuzzy set theory, Measurement Science & Technology, 11, 2000, pp. 430-435.

[8] R.E.Moore, Methods and Applications of Interval Analysis, 1979, SIAM Studies in Applied Mathematics.

[9] B. De Baets, A. Marková- Stupňanová, "Analytical Expressions for the Addition of Fuzzy Intervals", Fuzzy Sets and Systems 91, 1997, pp. 203-213.

[10] D. H. Hong, R. I. Ro, "The Law of Large Numbers for Fuzzy Unbounded Supports", Fuzzy Sets and Systems 116, 2000, pp. 269-274.

[11] H. Rommelfanger, T. Keresztfalvi, "Multicriteria Fuzzy Optimalization Based on Yager's Parametrized t-norm", Foundations of Computing and Decision Sciences, 16, 1991, No.2, pp. 99-110.