

## Weakly Defined Measurements

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Abstract: Measurement, in the wide sense, is defined as a process of empirical, objective assignment of symbols to attributes of objects and events of the real world, in such a way as to describe them. Strictly defined measurement is defined as measurement that conforms to the paradigm of the physical sciences. Weakly defined measurement is measurement in the wide sense, but which is not strongly defined. Strongly and weakly defined measurements are analysed and compared. They are compared with other forms of symbolic representation.

### INTRODUCTION

Measurement is an essential enabling tool of modern thought. It is the means by which we describe the world. Its application has been extended from the physical sciences and technologies, in which it had its origins, to other domains of knowledge, such as the social, behavioural and decision sciences. In our thinking we have adopted the programme of Galileo: to count what is countable, measure what is measurable, and to make measurable what is not measurable.

The requirements of those wider domains of knowledge and reasoning has led to the broadening of the concepts of measurement, so as to encompass other forms of objective symbolic description.

The development of computing which can handle symbolic information has given a further impetus to the development of the theory of symbolisation, and of the concepts of measurement widely defined.

The broadening of the concept of measurement has been disputed, and measurement in the physical sciences remains the normative paradigm. In any case, it is generally accepted that some forms of measurement, widely defined are stronger than others.

This paper deals with the aspects of more widely defined measurement, and the distinction between strongly and weakly defined measurement.

### HISTORICAL DEVELOPMENT

The history of the development of the concepts of measurement is given outline in [1], which cites underlying literature.

Recent years have mainly seen advances in the study of general representation of symbols. An outline of the current state of this topic is given in [2].

### DEFINITION OF MEASUREMENT

It is proposed here to adopt a wide definition of measurement, as a basis of discussion.

Measurement, in the wide sense, is thus defined as a process of empirical, objective assignment of symbols to attributes of objects and events of the real world, in such a way as to describe them [1,2].

The essential elements of this definition require to be analysed.

Firstly, measurement is defined as the assignment of symbols to properties of objects and events, and is thus the description of properties of objects or events and not of the objects or events. Measurement is based on a clear concept of a property, as an abstract aspect of a whole class of objects of which individual instances, or manifestations, are the subject of measurement.

The definition states that the assignment of symbols in measurement is such that the symbols describe the property of the object or event. The meaning of this can be explained as follows. Consider that a symbol, or measure, is assigned by measurement to the property of an object, and other symbols are assigned by the same process to other manifestations of the property. Then the numerical relations between the symbols or measures, in the symbolisation system adopted, imply and are implied by empirical relations between the property manifestations.

The next aspect of the definition of measurement that requires discussion, is the fact that measurement is an objective process. By this is meant that the numbers assigned to a property by measurement must, within the limits of error, be independent of the observer.

The informal definition of measurement presented above stresses the fact that measurement is an empirical process. This means first that it must be the result of observation and not, for example, of a thought experiment. Further, the concept of the property measured must be based on empirically determinable relations and not, say, on convention.

### PRAGMATICS OF MEASUREMENT

The adoption of a wide definition of measurement necessitates a comparison between widely defined measurement and other forms of symbolic description [2].

Firstly, measurement is free of the ambiguity and vagueness of other forms of description, say those of natural language.

Secondly, measurement is not mere naming, but provides information about the relation of a particular manifestation of a quality and other manifestations of the same quality.

Thirdly, and most significantly, the objectivity of measurement makes measures invariant and indisputable in logical discourse.

Finally, measurement is based on empirical observations and empirical relations. It thus provides strong knowledge.

These pragmatic strengths of measurement apply to widely defined measurement, as much as to measurement more strictly defined.

### STRONGLY AND WEAKLY DEFINED MEASUREMENT

As mentioned above measurement in the physical sciences is the normative paradigm, the model that all measurement strives to emulate.

Measurement in the physical sciences is based on:

- (i) precisely defined empirical operations,
- (ii) mapping on the real number line on which an operation of addition is defined,
- (iii) well formed theories for broad domains of knowledge.

Measurement that has the above characteristics may be termed strongly, or strictly defined.

Measurement that represents representation by symbols of properties of entities of the real world, based on an objective empirical process, but lacks some, or all, of the above distinctive characteristics of strong measurement, may be termed weakly defined.

### ELEMENTS OF THE FORMAL THEORY OF MEASUREMENT

#### Representation by numbers

The concept of measurement informally presented above will now be presented formally using the representational or model theory approach [2].

A representational theory of measurement has four parts:

- (i) an empirical relational system corresponding to a quality.
- (ii) a number relational system
- (iii) a representation condition
- (iv) a uniqueness condition.

These will now be considered.

- (i) Quality as an empirical relational system

A quality is a property of an object.

Consider some quality and let  $q_i$  represent an individual manifestation of the quality  $Q$ , so that we can define a set of all possible manifestations as

$$Q = \{q_1, \dots\}$$

Let there be on  $Q$  a family  $R$  of empirical relations  $R_i$

$$R = \{R_1, \dots, R_n\}.$$

Then the quality is represented by an empirical relational system

$$Q = \langle Q, R \rangle$$

- (ii) Numerical relational system

Let  $N$  represent a class of numbers

$$N = \{n_1, \dots\}$$

Let there be on  $N$  a family  $P$  of relations

$$P = \{P_1, \dots, P_n\}.$$

Then

$$N = \langle N, P \rangle$$

represents a numerical relational system.

Commonly  $N$  is just the real number line  $Re$ , with numerical relations defined on it.

- (iii) Representation condition

The representation condition requires that measurement be the establishment of a correspondence between quality manifestations and numbers in such a way that the relations between the referent property manifestations imply and are implied by the relations between their images in the number set. Formally, measurement is defined as an objective empirical operation  $M$

$$M: Q \rightarrow N \text{ so that } n_n = M(q_n).$$

such that  $Q = \langle Q, R \rangle$  is mapped homomorphically into (onto)  $N = \langle N, P \rangle$

The above homomorphism is the representation condition.

Firstly it implies that if  $q_n$  is related to  $q_m$  by an empirical relation  $R_{k_i}$ , that is  $R_{k_i}(q_n, q_m)$ ,  $P_k$  is the numerical relation corresponding to  $R_{k_i}$ ,  $n_n = M(q_n)$  is the image of  $q_n$  in  $N$  under  $M$  then  $R_{k_i}(q_n, q_m)$  implies and is implied by  $P_{k_i}(n_n, n_m)$

Measurement is a homomorphism, rather than an isomorphism, because  $M$  is not one-to-one, it maps separate but indistinguishable property manifestations to the same number.

$$S = \langle Q, Z, M \rangle$$

constitutes a scale of measurement for  $Q$ .

$n_n = M(q_n)$ , the image of  $q_n$  in  $N$  under  $M$  is called the measure of  $q_j$  on scale  $S$

- (iv) Uniqueness condition

The requirement that the fundamental measurement procedure of a scale should map the empirical relational system  $Q$  homomorphically into the numerical relational system  $N$  does not determine the mapping uniquely.

There is an element of arbitrary choice in the setting up of scales of measurement. For example in the case of scales based on additive combination, for instance, the choice of the unit standard is arbitrary.

The requirement of homomorphism thus defines a class of scales that may be called equivalent. The class of transformations, which transform one member of a class of equivalent scales into another, is called the class of admissible transformations. The conditions which admissible transformations must satisfy, are known as the uniqueness conditions.

### Representation by general symbols

Measurement as defined above can be seen as a special case of general representation of entities by symbols.

An object termed the symbol may be said to represent, or to carry information about another object or event termed the referent, by bearing a known relation to it.

We can now briefly present the formal theory of symbolic representation, which essentially follows the discussion of measurement above.

Let  $q_n$  be a referent entity. Consider further that  $q_n$  is a member of a family or set of similar entities  $Q$ ,

$$Q = \{q_1, \dots, q_n, \dots\},$$

$Q$  is termed the referent set. Let there be on  $Q$  a family  $R$  of relations  $R = \{R_1, \dots, R_n\}$ . We may term  $Q = \langle Q, R \rangle$  termed the referent relational system.

Let now  $z_n$  be a symbol entity. Consider further that  $z_n$  is a member of a family or set of similar entities  $Z$ ,

$$Z = \{z_1, \dots, z_n, \dots\},$$

$Z$  is termed the symbol set. Let there be on  $Z$  a family  $P$  of relations  $P = \{P_1, \dots, P_n\}$ . We may term  $Z = \langle Z, P \rangle$  the symbol relational system.

Let there be a mapping

$$M: Q \rightarrow Z$$

so that  $z_i = M(q_i)$ . Further let there be a mapping

$$F: Q \rightarrow Z$$

so that  $P_n = F(R_n)$ .

We may define  $C = \langle Q, Z, M, F \rangle$

where  $C$  is the representation code, and its inverse, the interpretation code.  $z_n$  is termed symbol of or for  $q_n$ .

### Language

For the purposes of this paper it must suffice to state briefly that a language is a collection of symbols, together with the rules of their combination, or syntax. in other words, a general symbol representation system.

Thus using the kind of notation employed above a language  $L$  for the description of objects of the real world and their relations, may defined as  $L = \langle Z, P, G \rangle$ , where  $Z$  is a set of individual symbols,  $P$  is a set of relation symbols and  $G$  is a set of function symbols.

A typical sentence in such a language is  $P_i(z_1, z_n)$ .

A code, such as  $C$  above, describing the correspondence of the linguistic symbols to the real world, or more generally to extra-linguistic entities and their relations, constitutes the semantics of the language.

### Theory

A theory  $T$  in a language  $L$  is a set of sentences in  $L$ .

A theory may be a representation, or model, of a domain in the extra-linguistic world under a set appropriate codes  $C_1, C_2, \dots, C_n$ .

## CHARACTERISTICS OF WEAK MEASUREMENT

Having defined weak measurement it is now proposed to examine characteristics of measurement defined as weak.

It is to be noted that representation by symbols that is not objective, empirical, or does not represent an empirical relational system on the quality, does not constitute measurement.

Weak measurement, has one or more of the following features:

- (i) it is based on an ill-defined concept of the quality
- (ii) there is significant uncertainty in the empirical relational system that it represents
- (iii) the symbolic relational system. has limited relations defined on it
- (iv) there is no adequate theory relating the measurement to other measurements in the same domain

The various aspects of weak measurement will now be considered in more detail.

### CONCEPT DEFINITION

The establishment of a scale of measurement of a quality must be based on a satisfactory concept of the measurand.

The formation of a quality concept is normally a recursive process. The establishment of a basic concept leads to a scale of measurement defined on it. Observations using the scale of measurement lead to a refinement of the concept and hence to a development of the scale. Measurement with a refined scale may ultimately lead to a rich theory of the domain, which establishes a definition of the concept and strong measurement scales.

An example of such recursive concept formation in the physical sciences is the development of the concept of temperature. Starting with a vague concept of "the degree of hotness of a body", scales of temperature based on practical instruments were developed. Such scales led to the refinement of the concept of temperature and improved thermometers. Observations with improved thermometers led to theories of heat and ultimately to an integrated theory of thermodynamics. It is on such an integrated theory of thermodynamics that the temperature scale is currently defined in physical science [3,4]

Problems of concept formation arise commonly in social science and psychology.

Consider for example the concept of poverty, which is much discussed in political discourse in natural language. It is extensively discussed in sociological literature, and is the subject of much sociological measurement. The development of the concept is clearly considered in [5] and [6], which cite more detailed literature. Sociologists distinguish between absolute and relative poverty. Absolute poverty occurs when people fail to receive sufficient resources to maintain health and efficiency, often expressed in terms of calories, or other nutritional measures. Relative poverty is defined by the general

standard of living in a society, and by what is culturally deemed as poor. The relative poverty of a household may be described by reference to some percentile of income distribution, which is deemed to represent the poverty line. Relative poverty thus varies between societies and for the same society over time. It is not possible here to consider the measurement of poverty in any detail, but it can be seen from the above that the definitions of poverty are theory-laden and hence measurement of poverty is weak in the sense defined above.

Another example of the problem of concept formation is the notion of intelligence. The term intelligence is much used in natural language discourse, but its definition is vague.[7] provides excellent introductions to the problem of the formation of the concept of intelligence, and cite more detailed literature. Many tests are available for measuring intelligence, but there is no certainty as to what intelligence is, or what intelligence tests are measuring. It may be that they are merely measuring the ability to score on the same form of intelligence test. The measurement of intelligence by intelligence tests, is an objective and empirical process, but is, in the sense defined above, a weakly defined measurement.

### UNCERTAINTY

All measurement, however strongly defined, is accompanied by uncertainty. However, in weakly defined measurement the uncertainty is different in kind.

Referring to the notation above we may distinguish between uncertainty in the mapping  $M$  and uncertainty in the empirical relational system  $R$ .

In strongly- or strictly defined measurement  $R$  is defined by a well-founded and well-formulated theory.  $M$  is again theoretically founded. Uncertainties are considered to be imperfections of the measurement process, and are estimated by corresponding analysis.

In weakly defined measurements, both  $R$  and  $M$  are based on empirical observations and empirical laws, in the determination of which there is uncertainty,

Good examples of uncertainty in weakly defined measurement are provided by psychophysical measurements of perceptions of sensory stimuli, such as brightness and loudness. The problem is reviewed in [7], which gives references to more detailed literature. It makes clear that human judgements of stimuli are poor, and that substantial uncertainty in both the concept and scale formation arises.

### STRENGTH OF MEASUREMENT SCALE

#### In strong measurement

In the classical approach to strictly, or strongly, defined measurement such measurement is confined to fundamental, or extensive measurement [1]. In such measurement, the empirical relational system in the quality  $Q$ , is  $R=(\approx, \prec, \circ)$ , consisting of an empirical partial order, and an empirical concatenation operation. The measurement operation  $M$

maps this  $Q$  into the real number line  $Re$ , with  $=, +$  defined on it.

For qualities for which this is not possible, we may define conjoint measurement structures, on the basis of empirical relational systems established for qualities conjointly [8].

Alternatively, it is possible to treat qualities that are not fundamentally measurable as derived variables [1]. The scales of measurement are then derived from an empirically established relation between two fundamentally measurable quantities, which maps manifestations of the variable into numbers, which have the same order as the manifestations have in the concept of the quality.

The approach to measurement adopted in the paradigm of the physical sciences is to establish a theory for a domain. The theory is mathematical. The concept of a variable is then defined by the theory. Measurement is an objective empirical operation, which maps manifestations of a variable into symbols of the theory.

#### In weak measurement

In weakly defined measurement, scales of measurement are classified in the literature according to the uniqueness conditions and their invariance under a class of transformations [1]. Usually scales of measurement are classified as nominal, ordinal, interval and ratio. The wider the class of admissible transformations of a scale, the narrower is the class of meaningful statements in terms of the symbols of the scale.

For example, in nominal measurement the quality  $Q$  has only an empirical indifference relation  $\approx$  on it. The symbol system  $Z$  is one in which the symbols  $Z$  may be very general and the symbolic relational system  $P$  consists only of the identity relation  $=$ . The scale is invariant under any one-to-one transformation. Statements implying order, or distance, are thus not meaningful in the scale.

An example of a nominal measurement is the classification of members of a population according to their first language, if it is objective and based on empirical observation. Such nominal measures may be used for empirical observations, relating first language to, say, income, religion and educational level. The class of meaningful statements in terms of nominal measures is limited. In statistics it is meaningful to state a mode of a distribution, but not a median, or a mean.

It should be noted that wide measurement admits a very wide class of empirical relations  $R$ , and correspondingly a variety of symbolic systems  $Z$ . There are, as yet, no measures developed for the strength of diverse wide symbolisation systems.

Staying with linguistic examples, it is possible to classify languages by empirical and objective criteria [9], in such a way that it is possible to express, say, a distance of one language from another. For example Polish is close to Slovak, more distant from Russian, yet more distant from German, and even more distant from Arabic.

## INADEQUACY OF THEORY FOR WEAK MEASUREMENT

In strictly or widely defined measurement the measure of a quantity is related to other measures on the same, or other, objects by a well formulated and extensive theory. For example measures of force on an object are related to measures of mass and acceleration.

In widely or weakly defined measurement such a theory is commonly lacking. For example, intelligence test scores may not be related with confidence with other measurements. To give an example from physical measurements, measures of hardness on the Mohs' scale [1], are not related by an adequate theory to other measures of properties of the object.

## SYMBOLISATIONS WHICH ARE NOT MEASUREMENT

It is useful for the sake of completeness to consider here assignments of numbers to properties in such a way as to describe them, but which are not measurements.

One example is classification of industries by a standard industrial classification. Assignments, on such a classification, of numbers to enterprises, are objective and based on an empirical observation. They represent the relation of the enterprise to others symbolised using the same classification. However, they are not measurement, to the extent that the system of classification is not based on an objective empirical system of relations of industries [10].

An important class of the descriptive assignment of numbers, the measurement status of which is problematic, arises in educational testing [11]. Marks in examinations may be objective, and are based on an empirical process, but it is problematic what they measure, other than the performance in a particular test. It is doubtful whether, when marks are treated as measures on a ratio scale, they are not in fact measures on an ordinal scale. This affects the meaningfulness of statistics on marks, such as t calculations of averages and the like. The conflation of marks, such as the calculation of weighted sums of marks, contains an element of subjectivity in the conflation scheme, which probably disqualifies such conflated marks from being considered measurements.

As a final example of descriptive numerical assignments that have doubtful measurement status, it is important to consider utility evaluation in formal decision making. Problems of decision sciences were an important stimulus to the development of wider concepts of measurement [12]. Formal concepts of utility, with considerable power, have been developed. However, it appears, that utility, however strongly defined, is only a description of the subjective judgement of the decision-maker. It is not a measure of any objectively defined characteristic of the object evaluated.

## MEASUREMENT AND NATURAL LANGUAGE

There is a strong relation between description by measurement, in the weak or wide sense, and description by natural language, which is in some of its functions a general form of symbolic representation.

It is not possible within the scope of this paper to consider natural language, even in outline. The barest skeleton of the comparison between measurement and description by natural language must suffice.

In measurement the meaning of a symbol is its reference. In natural language there are other views of meaning [13]. Meaning may be related to an idea in the mind of the originator or receiver of an utterance, or it may be considered as determined by conventional use.

The function of measurement is informational. Natural language has other functions such as aesthetic or phatic.

A linguistic symbol, even in its informational function may in addition to its denotation, convey other meanings, such as emotional colour.

As discussed above the essence of measurement is that it is an objective, empirical process. Description by natural language may be derived from empirical observation, though it is not necessarily so. It may be subjective, though it may have a high level of objectivity.

Description by natural language has often a degree of ambiguity, and vagueness. Description by measurement has generally a high degree of precision.

Finally description by measurement is generally more concise than description by natural language.

## CONCLUSIONS

The paper has presented the concept of widely defined measurement. It has presented the concepts of strongly and weakly defined measurement, and has analysed their respective characteristics.

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## REFERENCES

- [1] Finkelstein L., Theory and philosophy of measurement, In: P.H. Sydenham (ed) Handbook of Measurement Science, Chichester:Wiley,1982,pp.1-30.
- [2]Finkelstein L. , Foundational problems of measurement, in K. Kariya and L. Finkelstein, Amsterdam:2000 Ohmsha Press.
- [3]Preston T., The Theory of Heat, London: 1894, Macmillan & Co.
- [4] Michalski L., Eckersdor fK.and McGhee J., Temperature measurement, Chichester: 1991, Wiley.
- [5] Abercrombie N, Hill S., Turner B. , The Penguin Dictionary of Sociology, 4th Edition, London , 2000, Penguin Books.
- [6] Marshall G. (Ed), Oxford Dictionary of Sociology, Oxford, New York 1998, OUP.
- [7]Gregory R. L., The Oxford Companion to the Mind, Oxford, New York, 1987, OUP.

- [8] Krantz, D. H., Luce. R. D., Suppes, P., and Tversky, A. Foundations of Measurement, New York:1971, Academic Press.
- [9] Crystal D., The Cambridge Encyclopedia of language, Cambridge:1987, CUP
- [10] UK Office for National Statistics, UK standard industrial classification of economic activities, 1992, London: 1992, HMSO
- [11] Chase C. I., Measurement for educational evaluation, Reading Mass: 1974, Wesley
- [12] Roberts F. S., Measurement theory with applications to decisionmaking, utility and the social sciences, Reading Mass: 1979, Addison Wesley
- [13]Leech G., Semantics, Harmondsworth Mddx.: 1981, Penguin Books.