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The Optimal Correction of Known Measurement Windows

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ABSTRACT

Digital filters are commonly used in engineering. In many different technical problems it is necessary to apply digital filtration with required accuracy of signal processing in defined frequency domain with possible great attenuation of disturbances. This paper concerns correctors of known measuring windows like Hamming and Barthann according to requirements mentioned above.

Key words: low-pass filters, digital filter corrector, pass band, attenuation, frequency response.

1. INTRODUCTION

The digital filtration is widely used in the technical problems of signal processing because noise-free measurement can never be realised. There are methods working in the frequency and time domain, as well they can be realised in on- and off-line way. The methods basing on the weighted averaging can be considered as well as in the time and frequency domain and realised in on- and off-line way [2, 3]. The correction of averaging results extends the time of signal processing thus it seems to be useful in the off-line processing.

Let's assume that processing of the signal x(t) by the window with the weight function g(v), i.e. the window response y(t) is described by the formula [5]:

$$y(t_0) = \int_{-d}^{d} x(t_0 + v) \cdot g(v)dv \tag{1}$$

where the weight function (measurement window) *g(v)* fulfils the following conditions:

$$g(v) = g(-v), \quad \int_{-d}^{d} g(v)dv = 1,$$
$$g(v) = 0 \text{ for } |v| > d \tag{2}$$

where the interval (-d ,d) is the time width of the window. The Fourier spectrum of the window is described by the formula:

$$G(\omega) = 2 \cdot \int_{0}^{d} g(v) \cdot \cos(\omega \cdot v)dv \tag{3}$$

Expanding function $x(t_0+v)$ into Taylor series:

$$x(t_0 + v) = \sum_{i=0}^{\infty} \frac{v^i}{i!} \cdot x^{(i)}(t_0) \tag{4}$$

and denoting:

$$m_i = \int_{-d}^{d} v^i g(v)dv \tag{5}$$

we get the window response y(t) in the form:

$$y(t_0) = \sum_{i=0}^{\infty} \frac{m_i}{i!} \cdot x^{(i)}(t_0) \tag{6}$$

Since function g(v) is an even one, the odd moments are always equal to zero $m_{2i+1}=0$.
After expanding G(ω) and cosine functions into Taylor series the window spectrum can be formulated:

$$G(\omega) = \sum_{i=0}^{\infty} \frac{\omega^i}{i!} \cdot G^{(i)}(0) =$$
$$\sum_{j=0}^{\infty} \frac{\omega^{2 \cdot i}}{(2 \cdot j)!} \cdot (-1)^j \cdot 2 \cdot \int_{0}^{d} v^{2j} \cdot g(v) \cdot dv \tag{7}$$

Hence it appears, that even moments are equivalent to:

$$m_{2j} = G^{(2j)}(0)/(-1)^j \tag{8}$$

and it follows:

$$G(\omega) = \sum_{j=0}^{\infty} (-1)^j \cdot \frac{\omega^{2 \cdot j}}{(2 \cdot j)!} \cdot m_{2j} \tag{9}$$

Introducing the notation:

$$M_{2j} = \frac{(-1)^j}{(2j)!} \cdot m_{2j} \tag{10}$$

we get the relation:

$$G(\omega) = \sum_{j=0}^{\infty} \omega^{2 \cdot j} \cdot M_{2j} \tag{11}$$

where $M_0=1$ and M_{2j} are moments of the basic measurement window.

2. PRINCPILES OF CORRECTION

2.1. Construction of the corrector

If values y(t) are determined, then this window response can be corrected according to the formula:

$$y(t_0)_k = C_0 \cdot y(t_0) +$$
$$\frac{1}{2} \sum_{i=1}^{n} C_i \cdot [y(t_0 + T_i) + y(t_0 - T_i)] \tag{12}$$

and in the frequency domain it can be expressed by the relation:

$$y(\omega)_k = y(\omega)\cdot\left[C_0+\sum_{i=1}^{n}C_i\cdot\cos(\omega\cdot T_i)\right] \quad (13)$$

Such correction corresponds to the change of the basic window spectrum G(ω) to the form:

$$G(\omega)_z = G(\omega)\cdot G(\omega)_k =$$
$$G(\omega)\cdot\left[C_0+\sum_{i=1}^{n}C_i\cdot\cos(\omega\cdot T_i)\right] \quad (14)$$

Expanding cosine functions in Taylor series and using Eq. (11) we get:

$$G(\omega)_z = \sum_{j=0}^{\infty}M_{2j}\cdot\omega^{2j}\cdot\sum_{j=0}^{\infty}(-1)^j\cdot S_{2j}(C_i,T_i)\cdot\omega^{2j},$$

$$\text{where } S_{2j}(C_i,T_i)=\frac{1}{(2j)!}\cdot\sum_{i=0}^{n}C_i\cdot T_i^{2j} \quad (15)$$

The window, which frequency characteristic fulfils the condition $G(\omega)_z$=1 in the frequency range of the main lobe with the width ω_0, works as a low-pass filter. The corrected window as filter with maximum flat frequency characteristic in the pass band must fulfil the conditions:

$$G^{(2j)}(0)_z = 0 \text{ for } j=1,2,\ldots,k$$
$$G(0)_z = 1 \quad (16)$$

whence after consideration of the expression Eq. (15) the system of equations follows:

$$C_0+C_1+\ldots+C_n = 1/M_0$$
$$C_1\cdot T_1^2+C_2\cdot T_2^2+\ldots+C_n\cdot T_n^2 = 2!\cdot M_2 = A_2$$
$$C_1\cdot T_1^4+C_2\cdot T_2^4+\ldots+C_n\cdot T_n^4 = 4!\cdot\left(M_2^2-M_4\right) = A_4$$
$$C_1\cdot T_1^6+C_2\cdot T_2^6+\ldots+C_n\cdot T_n^6 =$$
$$6!\cdot\left(M_2^3-2\cdot M_2\cdot M_4+M_6\right) = A_6$$
$$\ldots \quad (17)$$

The moment M_0 is always equal to 1 if the basic window is normalised. The general form of the first equation of the above system admits the processing by the window that is not normalised. The normalisation follows through the correction which parameters were obtained on the base of this system.

Using the same window we can change frequency resolution, i.e. the main lobe width, by modifying its duration in time domain, i.e. parameter d. Therefore our consideration will be more general, if we now introduce the normalised frequency in the form:

$$\Omega = \omega\cdot\frac{2\cdot d}{\pi} \quad (18)$$

So constants A_{2j} must be multiplied by the expression:

$$\left(\frac{\pi}{2\cdot T}\right)^{2\cdot j} \quad (19)$$

The system of equations Eq. (17) is non-linear with unknowns C_i, T_i and it is not clear that for arbitrary n it has real solutions.

2.2. Determination of the corrector parameters

If we assume that time shifts Ti are known, then the system Eq. (17) will transform from non-linear to linear only with unknowns C_i. In this manner it is easy to resolve it with regard to C_i and the relations $C_i=f_i(T_1,\ldots,T_n)$ are known and in particular cases take the forms:

$$\text{for } n=1 \quad C_1 = A_2/T_1^2$$
$$\text{for } n=2 \quad C_1 = \left(A_2\cdot T_2^2-A_4\right)/\left(T_1^2\cdot T_2^2-T_1^4\right)$$
$$C_2 = \left(A_4-A_2\cdot T_1^2\right)/\left(T_2^4-T_1^2\cdot T_2^2\right)$$
$$\text{for } n=3 \quad C_1 = \left(A_2\cdot T_2^2\cdot T_3^2-A_4\cdot\left[T_2^2+T_3^2\right]+A_6\right)/\left(T_1^2\cdot\left[T_2^2-T_1^2\right]\cdot\left[T_3^2-T_1^2\right]\right)$$
$$C_2 = \left(A_2\cdot T_1^2\cdot T_3^2-A_4\cdot\left[T_1^2+T_3^2\right]+A_6\right)/\left(T_2^2\cdot\left[T_3^2-T_2^2\right]\cdot\left[T_1^2-T_2^2\right]\right)$$
$$C_3 = \left(A_2\cdot T_1^2\cdot T_2^2-A_4\cdot\left[T_1^2+T_2^2\right]+A_6\right)/\left(T_3^2\cdot\left[T_2^2-T_3^2\right]\cdot\left[T_1^2-T_3^2\right]\right) \quad (20)$$

The evaluation of parameters T_i can be done through maximising the criterion of the filter quality.

2.3. Criterions of the filter quality

The ideal window has wall spectral properties, i.e. the constant amplitude and the linear phase response in the pass band and zero response out of this band, no side lobes [2, 3]. Practical windows can only approximate this shape. We assume that the width of the main lobe of the basic window characteristic corresponds to the pass band. So we want to extend maximally the range of the main lobe, in which the response is linear with assumed accuracy. The next assumption regards to the rejection of disturbances and corresponds to the maximum level of the peak line of side lobes. We assume that this level will be as smaller as possible and at least not worse in comparison to the basic window. This compromise results from fact that the improvement of the signal recovering comes at a price of worsening the attenuation properties.

So the criterion of the filter quality should consist of at least two components [4]. The first one is associated with the condition, which determines the accuracy of the useful signal transfer in the possible width of frequency range within the main lobe of the characteristic:

$$\left|G(\Omega)_z-1\right|\le q \quad \text{for } \Omega<\Omega_{qz} \quad (21)$$

and corresponds to the ratio :

$$Q_1 = \frac{\Omega_{qz}}{\Omega_q} \quad (22)$$

where Ω_q is determined from the condition (21) for the basic window. The greater the above relation the better the correction.

Second component is an attenuation degree of disturbances. Because of the fact that the spectrum of noise very often is not known, we can assume that the power spectrum of disturbances is constant and we can only estimate the range of the disturbances band (Ω_1,Ω_2). So this component can be defined as the proportion of the greatness of an area included under the logarithmic spectral characteristic at Ω_1 to Ω_2 for the corrected and the basic window. The greater the proportion:

$$P = \frac{S_c}{S} = \frac{\int_{\Omega_1}^{\Omega_2} G(j\cdot\Omega)_z \cdot d\Omega}{\int_{\Omega_1}^{\Omega_2} G(j\cdot\Omega)\cdot d\Omega} \qquad (23)$$

the better the correction. Because of using logarithmic scale this criterion can be defined as relation:

$$Q_2 = 10^{P-1} \qquad (24)$$

The general quality criterion of the filter correction can be defined as:

$$Q = Q_1 \cdot Q_2 \qquad (25)$$

3. THE DOUBLE AVERAGING

Because of the fact that the improvement of the signal recovering comes at a price of worsening the attenuation properties we propose to use the double averaging with correction. The double averaging increases twice the attenuation according to the change of the basic window spectrum $G(\omega)$ to the form [1]:

$$G(\omega)_D = G(\omega)\cdot G(\omega) \qquad (26)$$

Whence the values of the minimum attenuation H and the averaged rate of the roll-off of the spectrum characteristic V are increased twice:

$$H_D = 2\cdot H$$
$$V_D = 2\cdot V \qquad (27)$$

Using Eq. (9) we get:

$$m_{2D} = 2\cdot m_2$$
$$m_{4D} = 2\cdot m_4 + \frac{4!}{2!\cdot 2!}\cdot m_2^2$$
$$m_{6D} = 2\cdot m_6 + \frac{2\cdot 6!}{2!\cdot 4!}\cdot m_4\cdot m_2$$
$$\ldots \qquad (28)$$

Therefore the attenuation improvement comes at price of the decrease the width of frequency range Ω_q, in which the signal recovering holds with determined precision. Hence it appears that the double averaging with correction is a good idea but the double corrected averaging can be the better solution.

4. DESIGN EXAMPLES AND RESULTS

The goal is to find optimal parameters of the corrected filter. The parameters C_i can be obtained as the unique solution of the system Eq. (17) for T_i determining maximum of the above-discussed criterion. To this effect we propose to use the direct search method to the optimisation process, because of the following reasons. Firstly, several local optima are in the search region (see Fig. 1). Secondly, the time shifts T_i must be the multiple of the measuring step, so the number of search in the determined seek region is finite.

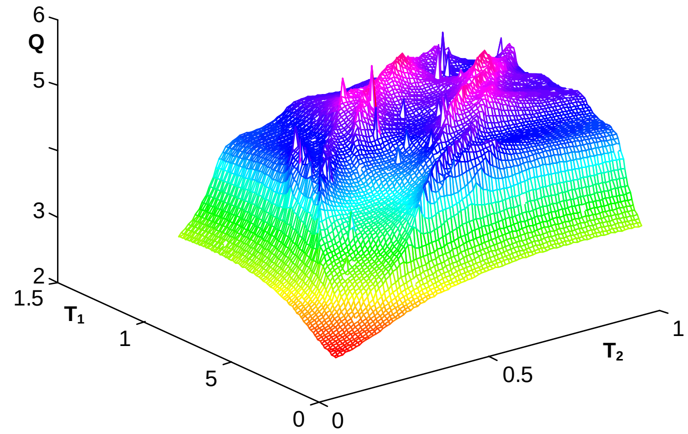


Fig. 1. The variability of the global criterion as function of time shifts $Q=f(T_1,T_2)$.

The algorithm has been implemented in Matlab environment. The calculations are realized for two windows: Bartlett-Hann and Hamming, for n=1,2,3 and the following conditions:

- The frequency range of disturbances was assumed from the frequency $\Omega_1=4$, corresponding to the main lobe width of the basic windows, to $\Omega_2=30$.
- The accuracy of the signal recovering was assumed q=0.01.

Because windows are defined for the same normalized frequency, then optimal results are independent from the resolution as well as the time duration of tested windows.

Fig. 1 is presented only as an example and illustrates the variability of global criterion as function of time shifts Q=f(T1,T2). One can see that the shape of this surface is symmetrical with respect to time shifts. The same regularity of course concerns the variability of the criterion in three-dimensional space, i.e. for three time shifts. The second component Q_2 of the criterion has the decisive influence on the distribution of criterion values Q in dependence of time shifts. The first component Q_1 has a more flat character. It appears that an application of such kind of correctors always increases bandwidth, but only specific parameters of the corrector are good on account of attenuation properties.

Firstly is considered the Bartlett-Hann window that is a linear combination of Bartlett and Hann windows with near side lobes lower than both Bartlett and Hann and with far side lobes lower than both Bartlett and Hamming windows. The main lobe width of this window is just the same like Bartlett and Hann window main lobes, i.e. $\Omega_0=4$.

Secondly is considered the Hamming window. This window has nonzero values for the end of averaging interval $g(-d) = g(d) \neq 0$. Therefore the spectrum of the corrected window possess the different shape at its beginning. This shape changes from short flat phase to maximum that can exceed the assumed accuracy. The greater the time shifts number n the greater the maximum. Because of this fact the correction with three time shifts for q=0.01 is completely unprofitable. So in case of windows like Hamming the level of the accuracy has an influence on the corrector parameters. In case of windows like Bartlett-Hann with the flat characteristic the accuracy changes only the value of criterion Q_1 as well as Q according to enlargement of Ω_q, without an influence on the corrector parameters. The differences between the shapes of the spectrum characteristics of tested windows with correctors in the range of the main lobe are shown in Fig. 2. The constant C_0 allows to move the characteristics up or down in order to increase of Ω_q.

The obtained parameters and values of the criterion and its components for correctors of both windows are presented appropriately in Table 1 and 2. The last two rows in these tables are respectively: the maximum filter response outside the main lobe - H [dB] and the averaged rate of the roll-off of the logarithmic characteristic $Y[\log(\Omega)] = 20\log\{G(\Omega)\}$ except for the main lobe - V [dB/dec]. These parameters have not been optimised. However H - the maximum peak side lobe level was increased, but other side lobe peaks are situated lower. So V – the side lobe roll-off is unchanged.

Table 1. Optimal parameters of corrected Bartlett-Hann window.

n	Bartlett-Hann window $g(v)=\frac{1}{d}\left[0.62-0.24\left\|\frac{v}{d}\right\|+0.38\cos\left(\frac{\pi\cdot v}{d}\right)\right]$				Double averaging with Bartlett-Hann window and its correction			Double averaging with corrected Bartlett-Hann window		
	-	1	2	3	1	2	3	1	2	3
C_0	-	1.486	1.4774	2.5599	1.5786	1.8627	8.0628	1.481	1.4724	2.5549
C_1	-	-0.476	0.7963	1.942	-0.5686	-1.4636	-9.8866	-0.476	0.7963	1.942
C_2	-	-	-1.2637	-1.3692	-	0.6109	4.0739	-	-1.2637	-1.3692
C_3	-	-	-	-2.1227	-	-	-1.2501	-	-	-2.1227
T_1	-	0.85	1.5	1.6	1.1	1.35	.7	0.85	1.5	1.6
T_2	-	-	1.3	1.7	-	1.8	1.4	-	1.3	1.7
T_3	-	-	-	0.8	-	-	1.75	-	-	0.8
Q_1	-	3.8487	5.2994	6.5887	2.8072	4.11	5.353	5.534	4.9984	5.9896
Q_2	-	0.8956	1.0086	0.9289	8.8554	9.6353	6.9198	9.8019	10.1725	7.2067
Q	**-**	**3.4468**	**5.3449**	**6.12**	**24.8584**	**39.6015**	**37.0418**	**54.2436**	**50.8465**	**43.1652**
Ω_q	0.2417	0.9302	1.2809	1.5925	0.6785	0.9934	1.2938	1.3376	1.2081	1.4477
S	1696	1614.8	1702.3	1575.4	3302.5	3364.6	3120.8	3377.3	3404.6	3150.7
H	-35.9	-30.5	-32.7	-21.5	-64	-66	-45	-62	-66	-43
V	-45	≈	≈	≈	-80	≈	≈	-75	≈	≈

Table 2. Optimal parameters of corrected Hamming window.

n	Hamming window $g(v)=\frac{1}{1.08\cdot d}\left[0.54+0.46\cos\left(\frac{\pi\cdot v}{d}\right)\right]$				Double averaging with Hamming window and its correction			Double averaging with corrected Hamming window		
	-	1	2	3	1	2	3	1	2	3
C_0	-	1.601	1.597	1.7567	1.7109	2.8045	3.6093	1.5963	1.593	1.7617
C_1	-	-0.5928	0.8174	-1.2997	-0.7209	2.6454	-3.9861	-0.5928	0.8174	-1.2997
C_2	-	-	-1.4154	0.8183	-	-4.4572	2.0822	-	-1.4154	0.8183
C_3	-	-	-	-0.2853	-	-	-0.7154	-	-	-0.2853
T_1	-	0.85	1.55	1.25	1.09	1.45	1.1	0.85	1.55	1.25
T_2	-	-	1.3	1.95	-	1.2	1.95	-	1.3	1.95
T_3	-	-	-	2.3	-	-	2.35	-	-	2.3
Q_1	-	4.3894	7.4479	4.6114	3.2221	5.3357	3.2759	3.8689	3.6903	3.3224
Q_2	-	0.8511	1.0978	0.967	8.3571	10.1377	9.5192	7.2621	9.4823	8.5598
Q	**-**	**3.736**	**8.1763**	**4.4593**	**26.9273**	**54.0917**	**31.1841**	**28.0962**	**34.9923**	**28.4389**
Ω_q	0.2251	0.9881	1.6765	1.0380	0.7253	1.2011	0.7374	0.8709	0.8307	0.7479
S	1365.4	1269.8	1420.7	1320.4	2624.4	2738.9	2701.6	2541.1	2699.3	2638.6
H	-44	-36	-34	-32	-69	-68	-67	-74	-68	-65
V	-18	≈	≈	≈	-37	≈	≈	-33	≈	≈

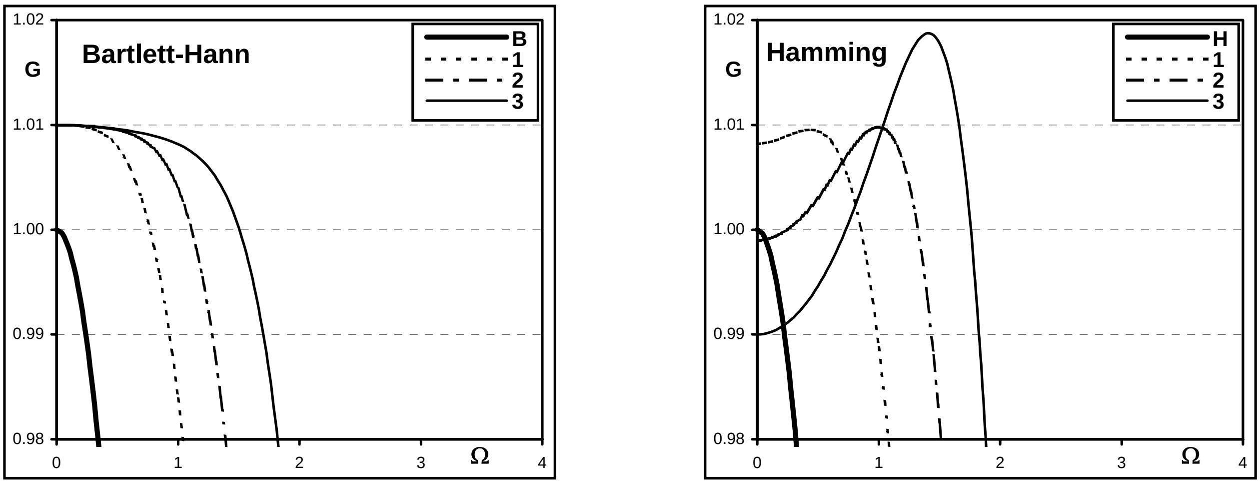


Fig. 2. The parts of spectra plots of tested windows with correctors for n=0,1,2,3 within the pass band.

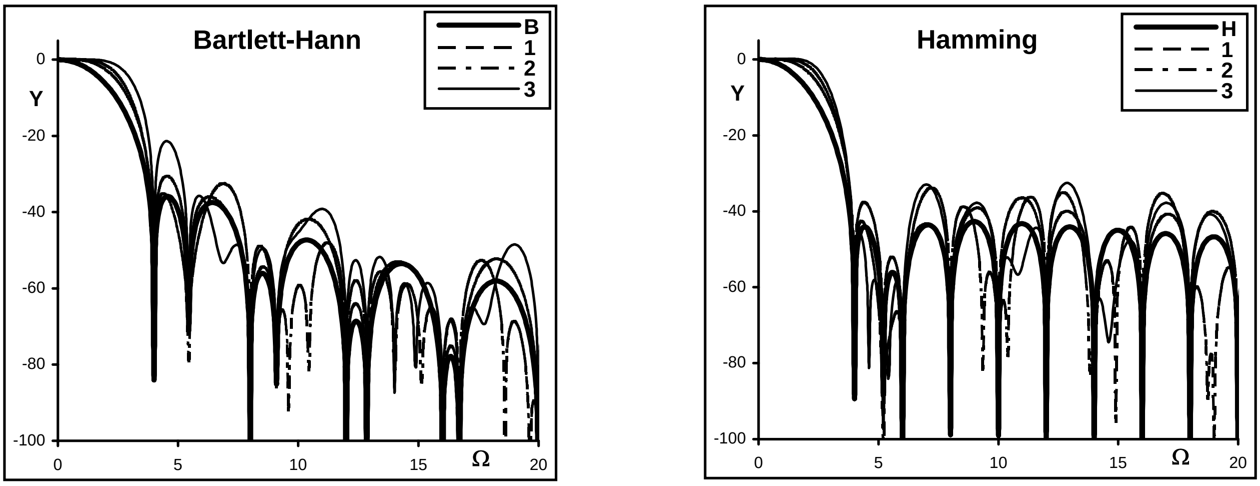


Fig. 3. The logarithmic spectrum characteristics of tested windows with correctors for n=0,1,2,3.

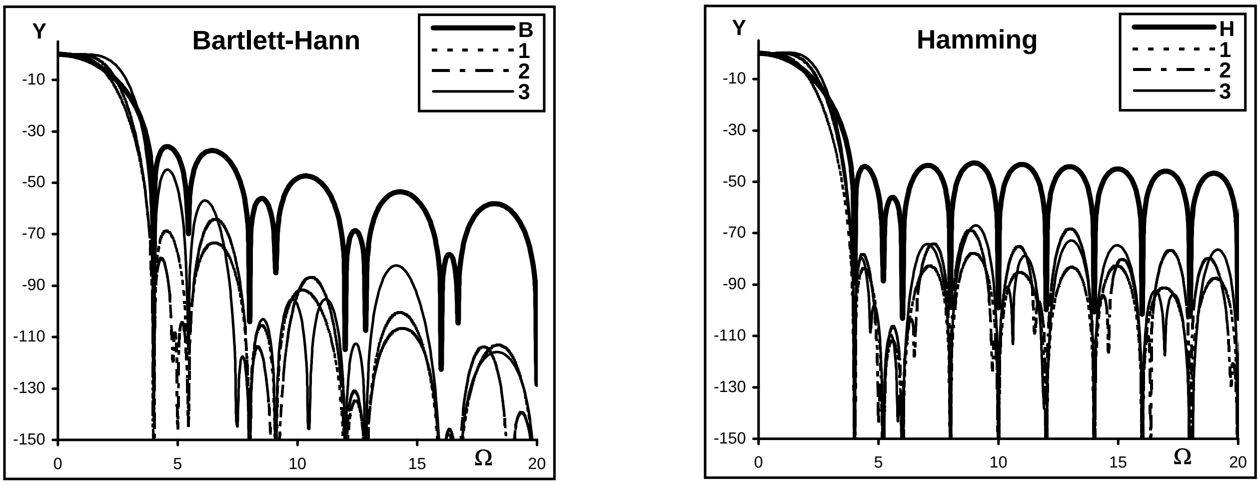
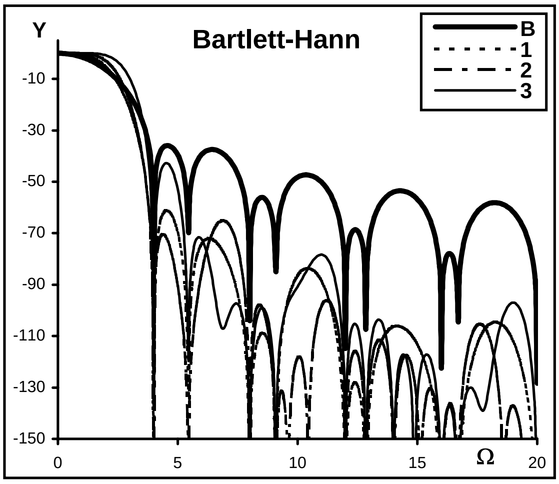


Fig. 4. The results of correction for double averaging.

The corrected windows with optimal parameters are presented in Fig. 3 in the form of the logarithmic spectrum characteristics. We can see that the enlargement of the pass band with assumed accuracy comes at a price of worsening the attenuation properties. The results for two time shifts correctors can be accepted as the best for both Bartlett-Hann and Hamming windows.

We propose also to use the double averaging and the correction of its results or the double averaging with corrected window. The proposition of these solutions is a good idea if only the time of the observation of the signal x(t) is sufficiently long. The obtained window has far better attenuation properties due to the double averaging and better recovery accuracy due to the correction. However the attenuation improvement comes at price of the decrease the width of frequency range Ω_q, in which the signal recovering holds with determined precision. Therefore the double corrected averaging is the better solution and the results confirm this opinion.



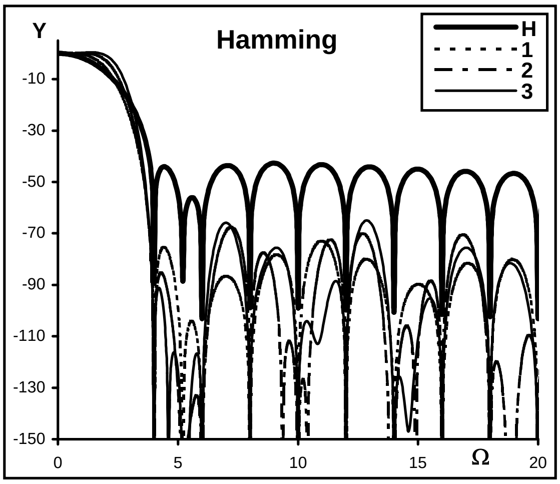


Fig. 5. The results for double averaging with corrected windows.

5. CONCLUSION

The results of optimisation of corrector parameters are promising. The application of this kind of correctors is a good idea in off-line processing. It always increases bandwidth with the assumed acurracy, but only specific parameters of corrector are good on account of attenuation properties. The double averaging with correction improves both the attenuation and recovering properties in comparison to the basic window, but decrease the width of frequency range Ω_q, in which the signal recovering holds with determined precision in comparison to the single averaging with the correction. Therefore the double corrected averaging seems to be the better solution than the double averaging with the correction. The values of quality criterions confirm this statement.

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