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The Method of Creating of Procedures for Numerical Differentiation of Signals

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ABSTRACT

The proposed method of creating of batch procedures for determining of signal derivatives bases on idea of so-called „ averaged differentiation". It is shown, that application of moments of weight function allows to design the correction procedures , which can be used for recovery of instantaneous values of derivatives on the base of „averaged" on certain time interval derivatives of noisy signal *x(t).* The problem of accuracy of proposed correction procedures, also in case of presence of disturbances influencing the processed signal *x(t),* is deeply discussed. The obtained formulae can be used for optimisation of procedures (minimisation of resultant influences caused by choice of width of interval of averaging and existence of disturbances). The optimisation of process of calculation of derivatives has been done for weight function in form of Nuttal's window. Furthermore, the possibility of radical simplification of proposed numeric procedures has been examined .

Keywords: Procedures for Signal Differentiation, Accuracy of Signal Differentiation, Measuring Windows.

1. INTRODUCTION

The presented idea of creation of numerical procedures for differentiation of signals (including non-typical ones) bases on method of so-called „average differentiation" [1]. Let us introduce the batch operation of differentiation of signal *x(t)* defined by :

$$x_g(t_0) = \int_{-d}^{d} x(t_0 + v) g(v) dv \tag{1}$$

where (-d, d) – interval of averaging, $g(\nu)$ - the normalised weight function fulfilling the condition :

$$\int_{-d}^{d} g(\nu) d\nu = 1 \tag{2}$$

If the weight function fulfils the additional conditions of the form:

$$g(\nu) = g(-\nu)$$

$$g^{(i)}(d) = g^{(i)}(-d) = 0 \quad \text{for} \quad i=0,1,2,....,k \tag{3}$$

then integrating by parts one can prove the following property:

$$x_g^{(i)}(t_0) = \int_{-d}^{d} x^{(i)}(t_0 + v) g(v) dv =$$

$$= (-1)^i \int_{-d}^{d} x(t_0 + v)\ g^{(i)}(v)\ dv \tag{4}$$

which holds for $i \le k+1$. It means, that averaged value of signal derivative $x_g^{(i)}(t_0)$ can be determined without necessity of differentiation of signal *x(t)* , because differentiation can be applied to known weight function $g(\nu)$ given in analytic form. It is very advantageous property, since differentiation of "real" signals , usually "contaminated" by disturbances, drastically decreases the level of useful signal in relation to the noise. The obtained result , i.e. $x_g^{(i)}(t_0)$, has to be treated as "averaged" derivative on interval $2d$. Thus, the necessary correction of result given by Eq.(4) has to be done in order to recover the value of derivative $x^{(i)}(t_0)$.

2. CORRECTION OF AVERAGED DERIVATIVE OF SIGNAL

Let us assume, that derivative $x^{(i)}(t_0)$ represents continuous and differentiable function and can be written in the form of Taylor's series:

$$x^{(i)}(t_0+v)=\sum_{j=0}^{\infty}\frac{1}{j!}v^j x^{(i+j)}(t_0) \tag{5}$$

Putting (5) to Eq. (4) we obtain:

$$x_g^{(i)}(t_0)=\sum_{j=0}^{\infty}\frac{1}{j!}x^{(i+j)}(t_0)\int_{-d}^{d}v^j g(v)dv \tag{6}$$

The element of the form:

$$m_j=\int_{-d}^{d}v^j g(v)dv \tag{7}$$

represents moment of j-order of weight function $g(v)$. If assumptions (2) and (3) hold, then

$$m_0=1,\quad m_{2k+1}=0,\quad m_{2k}\neq 0 \tag{8}$$

Thus, the expression (6) can be written in the form:

$$x^{(i)}(t_0)=x_g^{(i)}(t_0)-\sum_{j=1}^{\infty}\frac{1}{(2j)!}m_{2j}x^{(i+2j)}(t_0) \tag{9}$$

Using formula (9) for consecutive derivatives of order i, $i+2$, $i+4$,.....one obtains the following correction procedure :

$$x^{(i)}(t_0)=x_g^{(i)}(t_0)-\frac{1}{2}m_2 x_g^{(i+2)}(t_0)+$$

$$+(\frac{m_2}{2!}\frac{m_2}{2!}-\frac{m_4}{4!})x_g^{(i+4)}(t_0)-$$

$$-(\frac{m_2}{2!}\frac{m_2}{2!}\frac{m_2}{2!}-2\frac{m_2}{2!}\frac{m_4}{4!}+\frac{m_6}{6!})x_g^{(i+6)}(t_0)+-... \tag{10}$$

The coefficients expressed by moments quickly grow smaller . That is why only 2 or 3 initial addends of sum (10) can sufficiently approximate the value of derivative of i-order. Furthermore, the first addend from among neglected ones can be treated as rough evaluation of value of error of correction procedure. For example, on the base of (4) and (10) as well as Lagrange's idea on Taylor's series reminder we obtain:

$$x^{(i)}(t_0)\cong(-1)^i\int_{-d}^{d}x(t_0+v)\{g^{(i)}(v)-$$

$$-\frac{1}{2}m_2 g^{(i+2)}(v)+\frac{1}{4}(m_2^2-\frac{1}{6}m_4)g^{(i+4)}(v)\}dv$$

$$D_k\cong\frac{1}{8}(m_2^3-\frac{1}{3}m_2m_4+\frac{1}{90}m_6)|x^{(i+6)}(v)|_{mean} \tag{11}$$

where D_k – approximate error of correction procedure.

3. OPTIMISATION

If form of weight function $g(v)$ fits to the standard:

$$g(v)=\frac{k(r)}{d}\{f(\frac{v}{d}\}^r\quad,\ f(1)=f(-1)=0 \tag{12}$$

where $k(r)$ denotes the normalising coefficient, then we can easily prove the following property of moments:

$$m_{2k}=d^{2k}k(r)\int_{-1}^{1}x^{2k}f(x)^r dx-d^{2k}\Theta_1(2k,r) \tag{13}$$

where $\Theta_1(2k,r)$ – coefficient characterising the weight function. Putting (13) to expression (11) and considering, for example, only the two initial addends of sum (11) we obtain :

$$D_k\cong\frac{1}{4}\{(\Theta_1(2,r)^2-\frac{1}{6}\Theta_1(4,r)\}d^4|x_{meen}^{(i+4)}| \tag{14}$$

Let us consider , that processed signal contains random disturbances represented by power spectral density $G_z(\omega)$ Let $G(\omega)$ be the Fourier transform of measuring window $g(t)$. Using well known formula we can evaluate the mean square error P_z caused by disturbances :

$$P_z=\frac{1}{2\pi}\int_{-\infty}^{\infty}G_z(\omega)|G(\omega)|^2 d\omega\ \leq$$

$$\leq sup\ G_z(\omega)\int_{-d}^{d}(g(v))^2 dv$$

Putting (12) to above expression we obtain:

$$P_z\leq sup\ G_z(\omega)\frac{\Theta^2(i,r)}{d^{(2i+1)}} \tag{15}$$

where i – order of determined derivative , $\Theta(i,r)$ – the "new" coefficient depending on damping properties of weight function. Assuming the following criterion for optimisation:

$$E(d)=D_k+\sqrt{P_z} \tag{16}$$

one can determine the optimal interval (-d,d) for given form of weight function, if values of $|x_{mean}^{(i+4)}|$ and $sup\ G_z(\omega)$ are known.

The values of coefficients $\Theta_1(2k,r)$ and $\Theta(\iota,r)$ for weight function representing the Nuttal's window [2]:

$$g(v)=\frac{k(r)}{d}cos^r(\frac{\pi}{2}\frac{v}{d}) \tag{17}$$

are presented in Table 1. The superficial overview of data in Table 1 yields the following conclusion: if high order derivative is calculated, then it is necessary to use the weight function with sufficiently high r (especially, if described correction has to be done). For high values r

Table 1. Parameters of Nutal's window (see Section 3)

r	1	2	3	4	5	6
$k(r)$	$\frac{\pi}{4}$	1	$\frac{3\pi}{8}$	$\frac{4}{3}$	$\frac{15\pi}{32}$	$\frac{8}{5}$
θ(1,r)	1.2336	1.5708	1.9596	2.3216	2.6626	3.0071
θ(2,r)		4.9345	6.8909	9.3004	11.903	14.644
θ(3,r)			31.008	46.220	65.727	88.488
θ(4,r)				167.74	417.35	616.11
θ(5,r)					2951.1	4744.2
θ(6,r)						39613
$\theta_1(2,r)$	0.1894	0.1307	0.0994	0.0801	0.0669	0.0575
$10\ \theta_1(4,r)$	0.7872	0.4110	0.2502	0.1677	0.1200	0.0900
$10\ \theta_1(6,r)$	0.4288	0.1793	0.0907	0.0519	0.0323	0.0215
$100\ \theta_1(8,r)$	0.2691	0.0937	0.0404	0.0201	0.0111	0.0066

Table 2. The chosen results of experiment (see Section 4)

d [s^{-1}]	$x^{(1)}(0)$ $z=0$	$x^{(1)}(0)$ $z\neq 0$	$x^{(1)}(\pi/4)$ $z=0$	$x^{(1)}(\pi/4)$ $z\neq 0$	$x^{(2)}(0)$ $z=0$	$x^{(2)}(0)$ $z\neq 0$	$x^{(2)}(\pi/4)$ $z=0$	$x^{(2)}(\pi/4)$ $z\neq 0$
0.25	1.9999	2.026	0.00	0.0260	0.00	-1.599	-4.005	-5.605
0.50	1.9990	1.976	0.00	-0.0230	0.00	0.168	-3.999	-3.810
0.75	1.9947	1.995	0.00	0.0007	0.00	0.027	-3.988	-3.961
1.00	1.9817	1.982	0.00	0.00	0.00	0.001	-3.963	-3.962
1.25	1.9572	1.957	0.00	0.00	0.00	0.002	-4.009	-3.913
1.50	1.9158	1.915	0.00	0.00	0.00	0.00	-3.832	-3.831
1.75	1.8537	1.854	0.00	0.00	0.00	0.00	-3.707	-3.707
2.00	1.7682	1.768	0.00	0.00	0.00	0.00	-3.536	-3.536

we can observe the "avalanche" type of growth of coefficients $\Theta(\iota,r)$ for increasing values i and r . Thus, for efficient damping of disturbances one has to use wide interval of averaging. On the other hand, extending the width of interval of averaging we substantially enlarge the error D_k . Thus, we encounter contradictory requirements. Finally, we can state, that accuracy of determination of derivatives $x^{(i)}(t_0)$ of order higher than 2 in case of presence of intensive disturbances can not be high (small $E(d)$ seems to be unattainable).

4. RESULTS OF EXPERIMENT

The experiment has been carried out for Nuttal's window with r=5 and various width of averaging interval 2d. The first and second derivatives of noisy signal $x(t) = \sin(2t)$ have been determined. The mean square value of noise amplitude was $\sqrt{\bar{z}^2}$ = 0.026, the noise maximal amplitude was z_{max} = 0.07. The frequency interval of noise has been assumed as ω_z = (16 ......50) s^{-1}. The both derivatives have been determined for t_0 =0 and $t_0 = \pi/4$, under the presence of disturbances (noise) and , for comparison, in case of lack of disturbances. The calculation based on two initial addends of sum (11). The obtained results have been assembled in Table 2. The results of experiment can be easily explained. Considering data in Table 1 we can state, that for both cases, i.e. t_0 = 0 and $t_0 = \pi/4$, one can find the optimal width of averaging interval corresponding to value d_{opt} = 0.75 s^{-1}. The above optimal width , obtained for processing of noisy signal $x(t)$, minimises error of calculation of first derivative $x^{(1)}(t_0)$ as well as error referring to the second derivative $x^{(2)}(t_0)$. For $d < d_{opt}$ one can observe substantial errors caused by disturbances (noise). The disadvantageous influences of disturbances can be observed especially in case of calculation of the second derivative $x^{(2)}(t_0)$ for $d<d_{opt}$. In turn, for $d > d_{opt}$ errors are caused mainly by imperfect correction appearing through phenomenon of deformation of averaged derivatives, while influences caused by disturbances can be neglected. At least 6000 steps of integration procedure have been used in order to calculate the values of every derivative in Table 2. Thus, under the above circumstances the obtained results can be treated as satisfying.

5. THE SIMPLIFIED PROCEDURES OF NUMERICAL INTEGRATION

The idea of radical simplification of procedure of calculation of integral (11) seems to be tempting, because determination of derivatives would be brought to several , very easy manipulations on signal samples. To apply extreme simplification one can divide interval of averaging *2d* into 4 sections of identical width and use , for example, Simpson's method [4] for calculation of integral in formula (11). For above assumption set of values $x(t_0 - d)$, $x(t_0 - 0.5d)$, $x(t_0 + 0.5d)$, $x(t_0 + d)$ can be treated as the "input" data for calculation. It can be easily checked, that applying the above assumptions one obtains the following form of Eq. (11):

$$x^{(1)}(t_0) = \frac{1}{d}\{a_1[x(t_0+d)-x(t_0-d)]+a_2[x(t_0+0.5d)-x(t_0-0.5d)]\} \tag{18}$$

where values of coefficients a_1, a_2 depend on type of weight function *g(v)* and type of applied correction in formula (11). Denoting the mean square value of disturbances as $\bar{z}^2$ we can estimate the error caused by disturbances :

$$D_z = \sqrt{\bar{z}^2}\,\frac{1}{d}\sqrt{2(a_1^2 + a_2^2)} \tag{19}$$

The respective error caused by form of formula (18) can be estimate by means of expression:

$$D_0 = \frac{1}{3}d^2(a_1 + \frac{1}{8}a_2)x^{(3)}_{mean} + \frac{1}{60}d^4(a_1 + \frac{1}{32}a_2)x^{(5)}_{mean} + .. \tag{20}$$

where coefficients a_1, a_2 have to fulfil the following condition of normalisation:

$$2a_1 + a_2 = 1 \tag{21}$$

and $x^{()}_{mean}$ denotes the respective mean values of derivatives of signal *x(t)* referring to assumed interval of averaging 2*d*. Treating the sum of errors $D_z + D_0$ as criterion for optimisation we can determine the optimal value *d* as function of parameter a_1 (parameter a_2 depends on a_1 – see Eq. (21)). If we take only the first addend of sum (20) , then minimisation of $D_z + D_0$ yields $a_1 = 0$ and $a_2 = 1$ (condition for normalisation (21)). We can improve the procedure of differentiation putting to the formula (18) values $x(t_0\ \mu\ k_1 d)$ and $x(t_0\ \mu\ k_2\ d)$ for suitable k_1 and k_2 . Now , the condition of normalisation obtains the form:

$$2k_1\ a1 + 2k_2\ a_2 = 1 \tag{22}$$

The relatively simple calculations yield:

$k_1 = 0.4396$, $k_2 = 1.0$
$a_1 = 1.4099$, $a_2 = -0.2277$

$$d_{opt} = 3.13(x^{(5)}_{mean}\sqrt{\bar{z}^2})^{\frac{1}{5}}$$

$$(D_z + D_o)_{min} = 0.616((\bar{z}^2)^2 x^{(5)}_{mean})^{\frac{1}{5}} \tag{23}$$

Quite similarly one can obtain parameters for calculation of the second or higher order derivatives of signal *x(t)* for $t=t_0$. Respectively, the even or odd number of sections of interval of averaging *2d* has to be taken into account. It is obvious, that necessity of normalisation (21),(22),etc., makes, that any form of weight function *g(v)* can not be assigned to the method. That is why the problem ought to be defined in a different manner. This "new" procedure can be described as it follows: assuming form of weight function *g(v)* one should divide the averaging interval 2*d* into 4,8,16, ... (generally 2^n) sub-intervals of identical width. Then, for each *n* we can determine the normalising coefficient. Finally, basing on normalising coefficients one can look for such minimal *n*, that result of calculation of derivative $x^{(i)}(t_0)$ under the presence of disturbances is sufficiently accurate. Of course, the optimal interval of averaging should be determined like it has been advised in description of experiments presented in Section 4. The results of computer simulation are put to the Table 3.

Table.3. The results of computer simulations for $d = 0.75$, *q* – correction coefficient.

n	*q*	$x(t_0) = sin(2\ t_0)$		*q*	$x(t_0) = cos(2\ t_0)$	
		$x^{(1)}(0)$, z=0	$x^{(1)}(0)$, $z \neq 0$		$x^{(2)}(0)$, $z = 0$	$x^{(2)}(0)$, $z \neq 0$
2	0.9911	1.9502	2.0587	0.6030	-3. 5969	-3.6426
3	0.9545	1.9731	1.9367	0.7621	-2.8896	-3.0726
4	0.9883	1.9877	1.9887	0.8753	-3.4021	-3.3736
5	0.9981	1.9929	1.9938	0.9562	-3.7917	-3.7626
6	0.9997	1.9938	1.9945	0.9875	-3.9337	-3.9063
7	0.9999	1.9940	1.9947	0.9968	-3.9744	-3.9471

It results from data in Table 3 , that calculations of the first derivative yield almost correct values for $n \geq 4$ (i.e. if interval $2d$ is divided into at least 16 sub-intervals). To obtain acceptable level of accuracy during calculation of the second derivative we should assume $n \geq 6$ which means, that interval *2d* ought to be divided at least into 64 sub-intervals . Thus, we can draw the conclusion, that radical simplification of calculation of integral in formula (11) while the average differentiation method with correction of results is used seems to be impossible. However, simplification basing on approximately 100 sub-intervals can be still advised. The mentioned number of sub-intervals makes, that application of correction coefficient q is not necessary.

6. REFERENCES

[1] W. Kordylewski, J. Wach, " Uśrednione różniczkowanie zakłóconych sygnałów pomiarowych", *Pomiary – Automatyka - Kontrola ,* No. 6, 1988 (in polish).

[2] A.H. Nuttal , "Some windows with very good sidelobe behaviour", *IEEE Trans. on Acoustic, Speech and Signal Processing,* Vol. 29, No. 1, 1981, pp. 84-91.

[3] M. Boćkowska , *Odtwarzanie stanu wejscia toru pomiarowego na podstawie obserwacji stanu wyjscia z jednoczesną identyfikacjajego dynamiki w warunkach intensywnych, losowych zakłóceń,* Ph.D.Thesis , Technical University of Szczecin, 1998 , (in polish).

[4] R.L. Burgen, J. D. Faires, *Numerical Analysis,* Boston, PWS-Kent Publishing Company, Third Edition, 1985.