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METROLOGICAL ASPECTS OF NON-STATIONARY SIGNAL TRANSFORMATION FOR DYNAMIC SPECTRUM ANALYSIS

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Abstract – The non-stationary signal spectrum in the information measurement system (IMS) is based on two-dimension time-frequency transformation (TFT). Although the precision of the computer TFT provides the errors much less than a limit of practical significance, the transformation result relates not to uniquely determined point, but to some uncertainty cell, the size of which is determined by "metrological" characteristics (MC) of TFT. The cell may change its form in the time-frequency domain, but its limitations stays not smaller than limit minimum value that relates to resolution limit capability of TFT. In the paper, we give the MC estimation for one of widely used spectrogram TFT. The global (applicable to the whole time-frequency plane) and local MC (legible only near some definite point), and their correlation are given in the paper.

Keywords: Metrological characteristics, time-frequency transformation, spectrogram, cell of uncertainty.

1. INTRODUCTION

Limitedness of local stationary signal conception for IMS that is used for processing the data relating to controlled object (subsystem) requires the use contemporary analysis methods. Such methods include time-frequency transformations (TFT) group [1-4] that serves as a base for obtaining the dynamical spectrum of non-stationary signal. Metrological characteristics (MC) for TFT-procedure were not being estimated supposing that computer technology is capable to provide perfectly precise TFT implementation.

The TFT, realizing two-dimension mapping the signal *s(t)* into time-frequency domain *(t↔ω)* is strictly not one-valued (in definite degree). The result of mapping, TFT(t,ω) relates not to single point, but to some field, – rectangular cell with dimensions Δt; Δ ω:

$$t,\omega \rightarrow \left\{s(t); t \rightarrow TFT(t,\omega)\right\} \Rightarrow (t;\omega) \in \left[\Delta t;\Delta\omega\right] \quad (1)$$

That uncertainty interval is near, but not equivalent, to dispersing error, as it is not associated with TFT-computing precision. The cell [Δt; Δω] has the lowest boundary of its area (not equal to zero) [5-7]. In the limits of that boundary, the increase of frequency resolution can not be achieved without lowering the time resolution ($\Delta\omega \rightarrow \Delta\omega / k \Rightarrow \Delta t \rightarrow k\Delta t$). The limit values *Δt, Δω* are not dependent directly on signal type and its non-stationary form, therefore they are the metrological characteristics of TFT. It is worth wile to determine the dependence of uncertainty cell on *t* and ω values, i.e. on TFT-arguments, and estimate the global limit for uncertainty cell suitable for any point of TFT. Described conformity is peculiar to any one of TFT variety [3], including the special class of time-scale (wavelet-) transformations.

Method for obtaining the TFT MC is proposed for transformation typical form – spectrogram (module square of short-time Fourier transformation (STFT) and spectrogram (module square of wavelet-transformation). This one will be analyzed in the next paper.

2. METROLOGICAL CHARACTERISTICS FOR SIGNAL TFT (SPECTOGRAM) AND TRANSFORMATION WINDOW

Independence on signal level *s(t)* is achieved by special normalization of its energy in the time and frequency domain to zero:

$$\int |s(t)|^2 \, dt = 1 \qquad \int |S(\omega)|^2 \, d\omega = 1 \quad (2)$$

where S(ω) is Fourier transformation. Integration limits are $[-\infty\,;+\infty]$, if they are not indicated. Owing to such normalization, $|s(t)|^2$ and $|S(\omega)|^2$ acquire the sense of probability distribution density and serve for signal mean value and the dispersion calculation. Analogous normalization is used for time and frequency forms of window *h(t)*, relating to STFT. Square of normalized values serves as distribution for obtaining MC of window that is the element of TFT procedure.

For time and frequency forms of *s(t)* и *h(t)* we have used (convenient for analytical representing the method) amplitude-phase representation:

$$s(t) = A(t)\exp(i\varphi(t)); \quad h(t) = A_h(t)\exp(i\varphi_h(t));$$

$$S(\omega) = B\exp(i\psi(t)); \quad H(\omega) = B_H(t)\exp(i\psi_H(t)) \quad (3)$$

Introducing this characteristics and obtaining with their help the time mean square value for signal and window in time and frequency domain, – allows obtain uncertainty-cell-parameters estimation, and so estimate TFT-MC. Compact representation for resolution-cell bounds simulating TFT-MC for time form *(s(t), h(t))*, and for frequency form *(S(ω), H(ω))* is determined on the basis of equations:

$$\sigma^2_{t;s(t)} = \int (t - t_{av.})^2 \, |s(t)|^2 \, dt;$$

$$\sigma^2_{t;h(t)} = \int (t - t_{av.})^2 \, |h(t)|^2 \, dt \qquad (4)$$

$$\sigma^2_{\omega;S(\omega)} = \int (\omega - \omega_{av.})^2 \, |S(\omega)|^2 \, d\omega;$$

$$\sigma^2_{\omega;H(\omega)} = \int (\omega - \omega_{av.})^2 \, |H(\omega)|^2 \, d\omega. \qquad (5)$$

where the second index of dispersion shows the averaging function corresponding to (2).

Then, the time and frequency resolution limit for signal and window is determined by inequalities:

$$\sigma_{t;s(t)} \cdot \sigma_{\omega;S(\omega)} > 1/2; \; \sigma_{t;h(t)} \cdot \sigma_{\omega;H(\omega)} > 1/2 \quad (6)$$

It is well known that STFT is the Fourier transform of signal localized with window *h*: $s_t(\tau) = h(\tau - t)s(\tau)$, i.e. STFT is equal to:

$$\text{STFT} = S_t(\omega) = (2\pi)^{-1/2} \int s_t(\tau) \exp(-j\omega\tau) d\tau =$$

$$= (2\pi)^{-1/2} \int s(\tau) h(\tau - t) \exp(-j\omega\tau) d\tau \,,$$

and spectrogram (SP) is TFT module square of STFT, i.e. spectrum module square for time-localized signal: $SP(t,\omega)=|S_t(\omega)^2|$. Spectrogram is also normalized, $\iint |S_t(\omega)|^2 \, dt d\omega = 1$, because of signal energy normalization to zero (2). It represents the two-dimensional probability distribution of signal energy in time-frequency plane, and is noted further as *f(t, ω)*, *i.e.. f(t, ω)*$\equiv SP(t,\omega)=|S_t(\omega)^2|$, and more, symbol *f* will be keep (with clear-in-context sense) also for conditional (marginal) distributions, the argument (t or *ω)* of which will serve as a sense indicator:

$$f(t) = \int f(t,\omega) d\omega; \qquad f(\omega) = \int f(t,\omega) dt \qquad (7)$$

This one allows determine the mean value and dispersion of time and frequency for spectrogram. In particular, the expression for dispersion is of the form:

$$\sigma^2_{t;SP} = \iint (t - t_{av.;SP})^2 \, |S_t(\omega)|^2 \, dt d\omega$$

$$\sigma^2_{\omega;SP} = \iint (\omega - \omega_{av.;SP})^2 \, |S_t(\omega)|^2 \, dt d\omega \qquad (8)$$

where

$$t_{av.;SP} = \iint t \, |S_t(\omega)|^2 \, dt d\omega; \quad \omega_{av.;SP} = \iint \omega \, |S_t(\omega)|^2 \, dt d\omega.$$

The number of transformations using (8) and equations for $t_{cp.;SP}$ and $\omega_{cp.;SP}$ allows obtain relation of spectrogram-MC with signal and window MC into time and frequency domain:

$$t_{av.;SP} = t_{av.;s(t)} - t_{av.;h(t)};$$

$$\omega_{av.;SP} = \omega_{av.;S(\omega)} + \omega_{av.;H(\omega)} \qquad (9)$$

$$\sigma^2_{t;SP} = \sigma^2_{t;s(t)} + \sigma^2_{t;h(t)};$$

$$\sigma^2_{\omega;SP} = \sigma^2_{\omega;S(\omega)} + \sigma^2_{\omega;H(\omega)} \qquad (10)$$

Equations (8)-(10) determine the relation of MC's of TFT in the spectrogram form with analogous characteristics for signal and window.

3. GENERAL (GLOBAL) REPRESENTATION FOR METROLOGICAL CHARACTERISTICS OF TFT (SPECTROGRAM)

The use of (6) determining the relation of MC in the form of dispersion for signal and window (for time and frequency domain distribution) with analogous MC for spectrogram (10), permit to obtain relation of spectrogram MC that do not relate to single determined point of transformation (spectrogram) and are of general (global) type. By simple calculation, the equation (11) may be obtained that by using (6) gives the (Δt;Δω)-cell size estimation in the TFT-domain, that is basic inequality (12) for global MC:

$$\sigma^2_{t;SP}\sigma^2_{\omega;SP} = (\sigma^2_{t;s(t)} + \sigma^2_{t;h(t)})(\sigma^2_{\omega;S(\omega)} + \sigma^2_{\omega;H(\omega)}) =$$

$$= \sigma^2_{t;s(t)}\sigma^2_{\omega;S(\omega)} + \sigma^2_{t;h(t)}\sigma^2_{\omega;H(\omega)} +$$

$$+ \sigma^2_{t;s(t)}\sigma^2_{\omega;H(\omega)} + \sigma^2_{t;h(t)}\sigma^2_{\omega;S(\omega)} \qquad (11)$$

$$\sigma_{t;SP}\sigma_{\omega;SP} \geq 1 \qquad (12)$$

The expression (11) is sufficed, and it determines the dependence of MC product (MC-P) on signal and windows MC. As it is seen, (11) is not equivalent to simple product of MC.

4. LOCAL MC FOR TFT

Besides general (global for TFT as a whole) in is worth wile to analyze the correlations for MC being fair for specific point of TFT (spectrogram), that is for spectrogram coordinates *t* and ω, or for small neighborhood of that point. Such relations (and estimations for obtaining them) are termed the local ones. As before, the mean square estimations of uncertainty for TFT variable values are taken for MC. However, for obtaining local relation estimations of MS, the fractional (marginal) distributions of time and frequency here are used as TFT arguments, and also computational mean values, mean square scatterings and dispersions of that TFT-parameters (TFT_P).

Conditional mean values and TFT_P distributions (for *t* and *ω*) are estimated by equations:

$$\omega_{av.|t} = \frac{1}{f(t)} \int \omega f(t,\omega) d\omega; \quad t_{av.|\omega} = \frac{1}{f(\omega)} \int t f(t,\omega) dt$$

$$\sigma^2_{\omega|t} = \frac{1}{f(t)} \int \left[\omega - \omega_{av.|t}\right]^2 f(t,\omega) d\omega; \; \sigma^2_{t|\omega} =$$

$$= \frac{1}{f(\omega)} \int \left[t - t_{av.|\omega}\right]^2 f(t,\omega) dt \qquad (13)$$

where *f(t)* и *f(ω)* are partial (marginal on time and frequency) distributions that are expressed by equations:

$$f(t) = \int f(t,\omega) d\omega = \int |s(\tau) h(\tau - t)|^2 \, d\tau \qquad (14)$$

$$f(\omega) = \int f(t,\omega) dt = \int |S(\omega) H(\omega - w)|^2 \, dw \qquad (15)$$

Using (14) и (15) allows obtain expressions for marginal MC:

$$\sigma^2_{\omega|t} = \frac{1}{f(t)}\int[\frac{\partial}{\partial\tau}A(\tau)A_h(\tau-t)]^2\,d\tau+$$

$$+\frac{1}{2(f(t))^2}\int\int A^2(\tau_1)A^2(\tau_2)A_h^2(\tau_1-t)A_h^2(\tau_2-t)\cdot$$

$$\cdot\left[\varphi'(\tau_1)-\varphi'(\tau_2)+\varphi'_h(\tau_1-t)-\varphi'_h(\tau_2-t)\right]^2 d\tau_1 d\tau_2 \quad (16)$$

$$\sigma^2_{t|\omega} = \frac{1}{f(\omega)}\int\left[\frac{\partial}{\partial w}B(w)B_H(\omega-w)\right]^2 dw+$$

$$+\frac{1}{2(f(\omega))^2}\int\int B^2(w_1)B^2(w_2)B_H^2(\omega-w_1)B_H^2(\omega-w_2)\cdot$$

$$\cdot\left[\psi'(w_1)-\psi'(w_2)-\psi'_h(\omega-w_1)+\psi'_h(\omega-w_2)\right]^2 dw_1 dw_2 \quad (17$$

where according to (3), A, A_h and B, B_h are amplitudes in amplitude-phase representation of signal and window, and amplitudes in such representations of frequency forms for *s* and *w*; φ, φ_h are phases of signal and window, and ψ, ψ_h are phases of that variables in their frequency representation.

We may notice that everyone conditional dispersion is the sum of two components (addendums), which qualitatively differ one from another. The first addendum depends on phases and must be greater than zero. Therefore, multiplying MC of (16) and (17), there may be obtained inequality:

$$\sigma^2_{\omega|t}\sigma^2_{t|\omega} \geq \frac{1}{f(t)}\int\left[\frac{\partial}{\partial\tau}\left\{A(\tau)A_h(\tau-t)\right\}\right]^2 d\tau +$$

$$+\frac{1}{f(\omega)}\int\left[\frac{\partial}{\partial w}\left\{B(w)B_H(\omega-w)\right\}\right]^2 dw \quad (18)$$

For obtaining the MC basic inequality for TFT, it is convenient to introduce the definition of locally (in the point t,ω) normalized signal and spectrum values using equations:

$$\eta_t(\tau) = \frac{s(\tau)h(\tau-t)}{\sqrt{f(t)}} \quad -\textit{locally normalized signal}$$

$$\mu_\omega(w) = \frac{S(w)H(\omega-w)}{\sqrt{f(\omega)}} \quad -\textit{locally normalized spectrum} \quad (19)$$

where *f(t)* and *f(ω)* are given by (14) and (15).
Introduced normalization means the reduction of signal to unit energy (or unit norm in L^2):

$$\int|\eta_t(\tau)|^2\,d\tau = \int|\mu_\omega(w)|^2\,dw = 1 \quad (20)$$

Using the notations for normalized signal, we can obtain main local (i.e. for point *t*, ω as TFT-argument) inequality for MC of spectrogram:

$$\sigma^2_{\omega|t}\sigma^2_{t|\omega} \geq \int\left[\frac{d}{d\tau}|\eta_t(\tau)|\right]^2 d\tau\cdot\int\left[\frac{d}{dw}|\mu_\omega(w)|\right]^2 dw \quad (21)$$

5. CONNECTION OF GENERAL AND LOCAL TFT-MC (SPECTROGRAM CALCULATED WITH h-WINDOW)

Locally in point *t* the normalized signal is the reflection of signal being formed with the aid of window *h* (fixed in point *t*), and of considering the formed signal as a function of τ, and reduction of its energy to zero:

$$\eta_t(\tau) = [s(\tau)h(\tau-t)]/\sqrt{f(t)} \quad (22)$$

where *f(t)* is determined in (14).
Hence, considering such locally normalized signal (LNS) as a function of *τ*,, and assuming the localization moment *t* to be the parameter, the ordinary Fourier transformation on τ for LNS may be carried out as follows:

$$S_t(\omega) \equiv F_t(\omega) = \frac{1}{\sqrt{2\pi}}\int\exp(-j\omega\tau)\eta_t(\tau)d\tau \quad (23)$$

$|F_t(\omega)|^2$ is also normalized function of ω, and therefore it is some distribution of ω. So with $|F_t(\omega)|^2$ we can determine such MC as mean square spectrum width for locally normalized signal:

$$\int(\omega-\omega_{av.|t})|F_t(\omega)|^2\,d\omega=$$

$$=\frac{1}{f(t)}\int(\omega-\omega_{av.|t})|S_t(\omega)|^2\,d\omega=\sigma^2_{\omega|t} \quad (24)$$

Actually, this is conditional mean-square spread for frequency in spectrogram. So, it may be determined the mean and dispersion for locally normalized signal:

$$\tau_{av.|t} = \int\tau|\eta_t(\tau)|^2\,d\tau$$

local time of normalized signal (25)

$$\sigma^2_{\tau|t} = \int(\tau-\tau_{av.|t})^2\,|\eta_t(\tau|^2\,d\tau$$

$\sigma_{\tau|t}$ *– local duration of LNS* (26)

Since $F_t(\omega)$ *and* $\eta_t(\tau)$ are the pair of connected Fourier transformations, then the equations (24) и (26) give the inequality (analogically to (6)) for MC which are marginal for *t*, i.e. for signal with local normalization, related to moment *t*:

$$\sigma_{\omega|t}\sigma_{t|\omega} \geq 1/2 \quad (27)$$

This inequality for MC estimates the correlation of the TFT window width and frequency dispersion in the spectrum of locally normalized signal.

6. LOCAL EAQUATIONS FOR BAND WIDTH AND TIME DISPERSION (MEAN SQUARE DURATION) OF NORMALIZED SIGNAL

Extending described method for obtaining the general (global) and local relations for spectrogram-type TFT MC, we may introduce the determination of signal spectrum localization being implemented with the aid of window frequency form ($H(\omega)$). Localized at point ω spectrum is represented by equation:

$$S_{\omega}^{loc.}(w) = H(\omega - w)S(w) \quad (28)$$

It's normalized form $\mu_\omega(w)$:

$$\mu_{\omega}(w) = \frac{H(\omega - w)S(w)}{f(\omega)} \quad (29)$$

is termed as *local normalized spectrum* (LNSp).

Inverse Fourier transformation of LNSp (i.e. $FFT^{-1}[\mu_\omega(w)]$) has the form:

$$f_{\omega}(t) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{f(\omega)}} \int \exp(jwt)S(w)H(\omega - w)dw \quad (30)$$

Since $f_\omega(t)$ is also normalized (in L^2) to one, then it has the sense of distribution and may serve for estimating the LNSp-dispersion:

$$\sigma_{t|\omega}^2 \equiv \int (t - t_{av.|\omega})^2 |f_\omega(t)|^2 dt = \frac{1}{f(\omega)} \int (t - t_{av.|\omega})^2 |s_\omega(t)|^2 dt \quad (31)$$

where $t_{av.|\omega} = \int t \cdot f_\omega(t)dt$; $s_\omega(t) = FFT_w^{-1}[\mu_\omega(w)]$.

Actually, (31) is the conditional time dispersion for given local frequency ω of spectrogram. As it is pointed out higher, on that basis it may be obtained the inequality:

$$\sigma_{\omega|\omega}\sigma_{t|\omega} \geq 1/2 \quad (32)$$

where marginal conditional dispersion of time and frequency are naturally determined for locally normalized spectrum:

$$\sigma_{\omega|\omega}^2 = \int (w - w_{av.})^2 \, |\mu_\omega(w)|^2 \, dw$$

$$\sigma_{\omega|\omega} - local \quad band \quad width \quad (for \quad LNSp) \quad (33)$$

$$w_{av.} = \int w |\mu_\omega(w)|^2 \, dw - mean \ frequency \ in \ LNSp \quad (34)$$

$$\sigma_{t|\omega}^2 \equiv \int (t - t_{av.|\omega})^2 \, |s_\omega(t)|^2 \, dt$$

$$\sigma_{t|\omega} - duration \quad of \quad LNSp \quad (35)$$

7. CONCLUSION

Despite of extremely high precision of modern computer that practically eliminate errors of TFT-implementation for non-stationary signals in IMS, the nature of this transformations based on the property called uncertainty principle in quantum physics. From the principle it is follows impossible to get desired high limit of precision simultaneously for two of certain type connected variables. In signal processing such variables are time and frequency, or time and scale, for wavelet transformations. It is well known that for traditional methods of signal analysis, there is the usual dilemma of necessary compromise between time and frequency resolution in different forms of spectral analysis. Creating new methods for non-stationary signal dynamical spectral analysis (two dimensional time-frequency and wavelet-transformations of different kinds) furthers to sharp growth of accuracy, and extending the interpretation abilities for results of signal transformation in IMS. In this connection, the results of signal transformation in IMS were metrologically analyzed for sensors and signal initial transformation only. It was supposed implicitly that signal transformation in computer on the basis of complex algorithms provides low precision for practical requirements. However, the TFT theoretic analysis shows that classical dilemma of ambivalence in requirements to precision of time-frequency (so as time-scale) transformations remain its significance.

For absolute accuracy signal processing by TFT it in necessary to find uncertainty element what point (in t,ω-plane) relate the result. The problem is most clear when it concerns the widely used method of short-time Fourier transformation (STFT). The choice of small transformation window (for better time localizing the signal frequency components) leads to the case when the spectrum itself will be calculated with big error because of shortness of signal segment used for its estimation. On the contrary, the choice of long-duration window results in increasing frequency resolution but time localization of spectrum becomes worse. However the estimations of such metrology aspects were of qualitative type.

The possibility of quantitative metrological estimations for modern methods of dynamical spectrum analysis on the basis of TFT has great signification. In direction of described problem we have proposed the analysis for conformity to physical laws for TFT errors formation in the case of spectrogram (STFT module square). The analysis method basis is of principally general character and is applicable to other TFT types that in the present do not find sufficiently wide application. The method is used time and frequency forms of signal and transformation window so as their amplitude-phase representation. Analytical expressions are shown for first and second statistical moments of time and frequency on the basis of distribution functions generated by the signal and window in their time-frequency representations (8 forms of moments). Analogous moments are carried out using distribution on the basis of spectrogram. Interrelations are shown that determine correlation of this two moment groups.

As metrological characteristics, the dispersions of TFT parameters are used. The lower limits for MC product are proposed. On this level the accuracy measurement of one TFT parameters (time or frequency) is provided only by the cost of inverse-proportional precision lowering for another parameter. Obtained equations determining lower limit of MC product and supplying uncertainty cell estimation for TFT result localization are analogous to uncertainty principle in physics, and reflect TFT nature itself.

Interrelations for MC are constructed on the two levels: *general or global relations of MC* applicable for estimations in any point of signal dynamical spectrum parameters, and *local MC relations* applicable and *adapted for specific determined point or not-great area of time-frequency plane representing the dynamical spectrum.*

The method of bivalence interrelation construction for TFT MC is based on formalized terms of locally normalized signal, conditional probability distributions (marginal on time and on frequency), that are formalization procedure on the basis of locally normalized *signal* and locally normalized *signal spectrum*.

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APPENDIX

The form of using the basic relation estimations between MC's of considered TFT spectrogram may be shown on some special applications.

- For signal *s(t)* and window *h(t)* in the form (1A) and (2A) the estimation of duration and frequency band may be represented by equation (3A):

$$s(t)=\left(\frac{\alpha}{\pi}\right)^{1/4}\exp\left[-\frac{\alpha t^2}{2}+j\beta t^2+j\omega_0 t\right] \qquad (1.A)$$

$$h(t)=(a/\pi)^{1/4}\exp\left(-\frac{at^2}{2}+jbt^2+j\omega_1 t\right) \qquad (2.A)$$

$$\left(\sigma_{t|s(t)}\right)^2=1/2\alpha;\qquad \left(\sigma_{\omega|S(\omega)}\right)^2=(\alpha^2+\beta^2)/2\alpha;$$

$$\left(\sigma_{t|h(t)}\right)^2=1/2a;\qquad \left(\sigma_{\omega|H(\omega)}\right)^2=(a^2+b^2)/2a \qquad (3.A)$$

Hence, taking into account the expression (10) in the base text, we can represent the general (global) TFT-MC product:

$$\left(\sigma_t^{SP}\right)^2\left(\sigma_\omega^{SP}\right)^2=\frac{1}{4}\left(\frac{1}{\alpha}+\frac{1}{a}\right)\left(\alpha+a+\frac{\beta^2}{\alpha}+\frac{b^2}{a}\right)=$$

$$=1+\frac{(\alpha-a)^2}{4\alpha a}+\frac{1}{4}\left(\frac{1}{\alpha}+\frac{1}{a}\right)\left(\frac{\beta^2}{\alpha}+\frac{b^2}{a}\right) \qquad (4.A)$$

Since all addendums in (4A) are greater than zero, so actually (in accordance with (12)) the inequality (5A) is:

$$\left(\sigma_t^{SP}\right)^2\left(\sigma_\omega^{SP}\right)^2\geq 1 \qquad (5.A)$$

- For signal in the form of (6A) and window in the form of Gaussian (7A):

$$s(t)=\sqrt{\frac{\alpha^{2n+1}}{(2n)!}t^n}\exp\left[-\frac{\alpha t}{2}+j\omega_0 t\right] \qquad n\geq 1 \qquad (6.A)$$

$$h(t)=s(t)=(a/\pi)^{1/4}\exp(-at/2) \qquad (7.A)$$

we have particular metrological characteristics:

$$\left(\sigma_{t|s(t)}\right)^2=1/2\alpha;\qquad \left(\sigma_{\omega|S(\omega)}\right)^2=\alpha/2;$$

$$\left(\sigma_{t|h(t)}\right)^2=(2n+1)/a^2;\quad \left(\sigma_{\omega|H(\omega)}\right)^2=a^2/4(2n-1) \qquad (8.A)$$

Then signal and window MC product (9A) and using the expression (11) in basic text, allow determine the product of MC for spectrogram in the form of equation (10A):

$$\left(\sigma_{t|s(t)}\right)^2\left(\sigma_{\omega|S(\omega)}\right)^2=\frac{1}{4};$$

$$\left(\sigma_{t|h(t)}\right)^2\left(\sigma_{\omega|H(\omega)}\right)^2=\frac{1}{2}\sqrt{\frac{2n+1}{2n-1}} \qquad (9.A)$$

$$\left(\sigma_t^{SP}\right)^2\left(\sigma_\omega^{SP}\right)^2=\frac{1}{4}+\frac{1}{4}\frac{2n+1}{2n-1}+$$

$$+\frac{a^2}{8\alpha}\frac{1}{2n-1}+\frac{\alpha}{2a}(2n+1) \qquad (10.A)$$

Minimum value of last two terms of (10A) is achieved if (a^2/α) will be equal to $\left(a^2/\alpha\right)=\sqrt{(2n+1)(2n-1)}$. This results in that optimum of all expression (10A) is given by (11A):

$$\mathrm{s}\,Min\left[\left(\sigma_t^{SP}\right)^2\left(\sigma_\omega^{SP}\right)^2\right]=(1/4)\left(1+\sqrt{\frac{2n+1}{2n-1}}\right)^2 \qquad (11.A)$$

It is seen from (11A) that if *n* is not great then minimum is slightly greater than 1. And (11A) achieves its lowest value for great *n*, when it actually is equal to one.

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