10th IMEKO TC7 International Symposium
June 30–July 2, 2004, Saint-Petersburg, Russia

ADVANCES IN THE MATHEMATICAL MODELLING OF SENSORS AND ACTUATORS

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Abstract – Mathematical modelling is a key enabling tool in the analysis and design of measurement systems and measuring instrumentation. This paper highlights the importance and the use of advanced mathematical modelling techniques for design, analysis and performance evaluation of sensors and actuators in measurement and instrumentation systems. After a brief introduction, it illustrates the use of numerical finite element (FE) modelling technique for analysis and design of sensors and actuators. This is based on two case studies in which FE modelling methodologies have been used for the analysis and design of a high performance torque motor actuator and a robust capacitive sensor used in this actuator for angular position sensing. Various modelling considerations are discussed and the methods of their FE realisation are analysed.

Keywords: mathematical modelling, sensors, actuators

1. INTRODUCTION

The mathematical modelling of systems and its application in measurement system description, analysis and design forms a core topic of measurement science [1, 2].

As a result of advances in technology, information processing in modern measurement systems is substantially implemented by standard information technology hardware and software. These components and processes of measurement systems are modelled and analysed using the methods of information technology, using models that are essentially functional. Modelling and analysis based on these methods have received much attention in recent literature of measurement science.

However the sensors, actuators, the object under measurement and its interaction with the measuring system present different modelling problems. They operate by the transformation of physical variables by the instrument element. The transformation must be modelled in terms of the physical embodiment of the element, that is its geometry and material properties as well as the physical laws that apply to it.

Idealised models are extensively used for system description, analysis of working principles, and conceptual design. However, detailed design requires realistic models. Such models have been developed and applied for a variety of sensors and actuators [2].

Recent advances in computer based modelling, for example in computational electromagnetics, have added powerfully to the capability of design and analysis of sensors and actuators. This is illustrated by the examples of modelling and design of a capacitive angular position sensor and a high performance torque motor actuator presented in this paper.

2. MATHEMATICAL MODELLING OF SENSORS AND ACTUATORS

It is now widely accepted that detailed analysis and design of sensors and actuators requires models which relate their detailed geometry, dimensions and material properties to their functional behaviour. In general, instrument transducers and sensors are often characterized by complex geometries and distributed properties. The physical laws governing their behaviour are represented by partial differential equations, which are often non-linear and transcendental. Analytical solutions of such realistic models are generally not feasible. However, the numerical finite element (FE) techniques have made possible the formulation and analysis of such models. It is in this area that significant progress is being made. The following sections below illustrate such models by two recent applications.

2.1. Capacitive sensors

Capacitive sensors are extensively used in many applications for measuring displacement, pressure, torque, flow, and other physical quantities [3-5]. In all these cases the primary sensor is based on the well-known capacitive technique in which the capacitance in a system of electrodes is changed owing to the redistribution of electric field caused by changes in the dielectric properties and/or geometric parameters of the sensor. For modelling, design and performance evaluation of these sensors as sub-systems, the core activities focus on the accurate computation and characterisation of 2D/3D electrostatic fields in complex geometry. Fig. 1 shows such a capacitive sensor used for angular position sensing in high-speed torque motor actuators shown in Fig. 2. The complex problems of analysis and design of such sensors can only be tackled by advanced mathematical methods

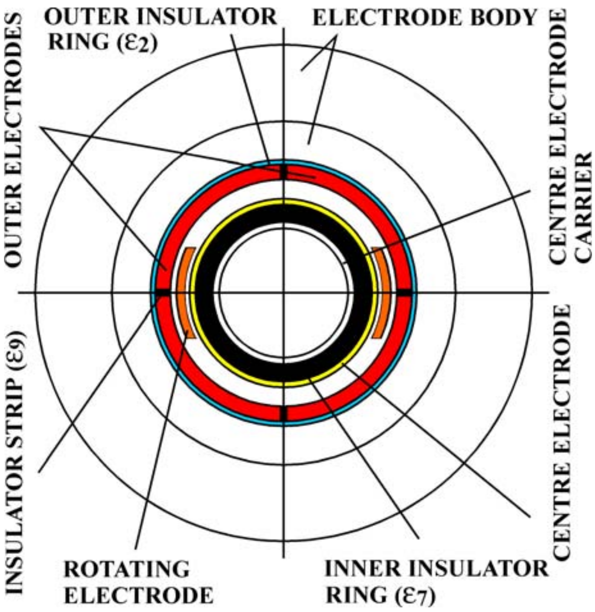

Fig. 1. Four-electrode capacitive sensor for angular position sensing in torque motor actuators.

based on, for example, numerical techniques such as the finite element method (FEM). In most cases, this means the numerical solution of the following Laplace's or Poisson's equation governing the field distribution in the 2D/3D problem domain Ω:

$$\begin{matrix} \nabla \cdot \varepsilon \nabla \Phi = 0 \\ \nabla \cdot \varepsilon \nabla \Phi = \rho \end{matrix} \quad \text{in } \Omega \qquad (1)$$

The numerical FE technique used to solve the above equations under given boundary conditions comprises a powerful mathematical modelling technique that allows the design, optimisation and performance modelling of such capacitive sensors [6].

2.2. Electromagnetic actuators

Electromagnetic (EM) solenoid actuators are widely used in many applications where repetitive, often high-speed, linear or rotating motions are required. Fig. 2 shows the cross section of such an actuator used for frame scanning in infrared thermal imaging devices. It is a limited angle torque motor actuator, which, unlike conventional electrical motors, executes 'oscillatory' limited angular deflections with a given frequency. These actuators are also ideally suited for aerospace servo-valve control and other similar applications. For normal operations, the actuator is fed by a 50-Hz sawtooth waveform which it is required to follow to deliver the required 50-Hz scanning frequency (image frame rate) of the mirror attached to one of the shaft ends with a maximum angular deflection of ±8°.

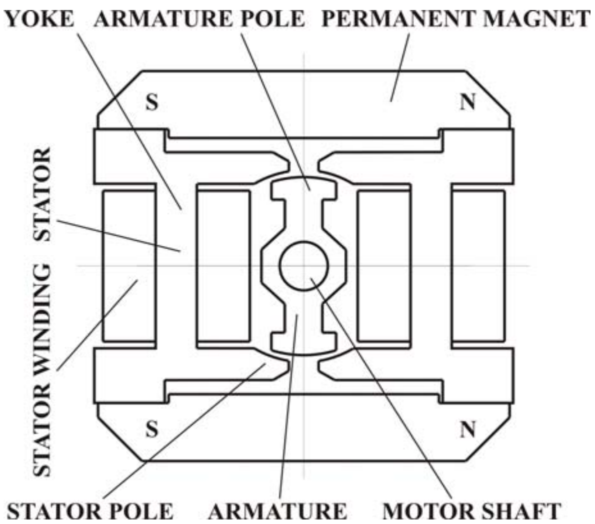

Fig. 2. Cross-section of a high-speed limited-angle torque motor actuator.

Although EM actuators are well covered in the literature [7-9], the whole area of modelling and analysis of this and other high frequency EM actuators (e.g. EM ejector valve used in sorting machines) is relatively new. The performance of such actuators is often dictated by complex electromechanical processes which have to be understood, analysed and predicted. They rarely operate in the steady state and various operational factors like start-stop duty, operating frequency, response time and damping have a significant influence on their design. The EM part of the system is represented by electric and magnetic circuits with self-inductance, resistance and reluctance which are subject to variations, in general, by eddy currents, saturation conditions, motional electromotive force (e.m.f.), demagnetisation and hysteresis. The mechanical part is represented by friction, damping, elasticity and inertia as well as external forces. Together these two parts form an equivalent electromechanical system which has to be optimised against the performance requirements and the complex behaviour of which is subject to static and dynamic analyses of these actuators. For high frequency EM actuators the thermal problem of temperature distribution and heat dissipation is of vital importance. Like all other electrical devices, they generate losses (e.g. ohmic loss in the winding, core and any circulating-current losses, etc.) manifested in the form of heat and temperature rise which may be considered to be one of the dominant factors in limiting the life of most high-speed actuators. Temperature-rise may significantly increase the winding resistance, impairing control. The frequency response may be altered because both the electrical and the mechanical time constants are temperature sensitive. Often tightly packed actuators may create serious problems of heat sinking from individual ones. Also the diverse combination of materials used and the very extensive duty cycles may give rise to myriad thermal problems unique only to high-speed actuators. The solutions to these problems may have a direct impact on the cost, size, reliability and feasibility of a given design. Compared to conventional solenoid actuators, the nonlinear and transient EM, thermal, and motional problems that need to be solved in high-speed actuators pose substantial challenges because of their

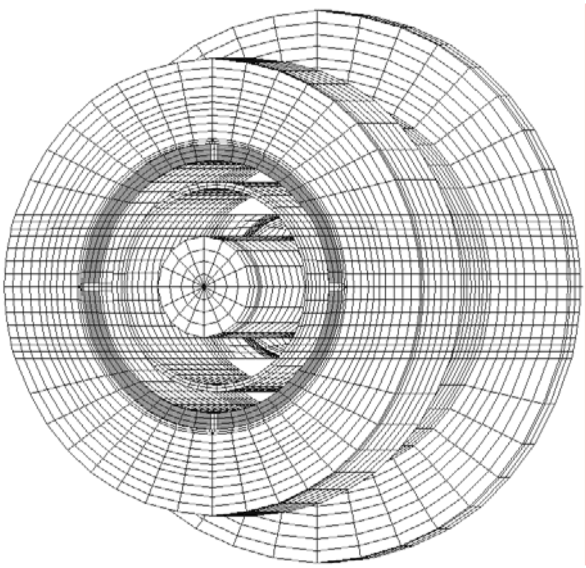

Fig. 3. 3D finite element model of the four-electrode capacitive sensor shown in Fig. 1.

high frequency of operation and the requirement for a continuous and fail-safe duty cycle.

In general, the mathematical model of the above electromechanical system can be adequately represented by the following four differential equations: (1) an electrical circuit equation for the excitation coil and control circuitry, (2) a nonlinear magnetic field equation (Poisson's equation) for the flux, the change of which changes the EM energy storage in the system and produces the magnetic force, (3) a mechanical equation for this force, load (e.g. pneumatic force), friction, inertia, acceleration, speed and displacement, and (4) a nonlinear thermal diffusion equation for the conduction of heat produced by electrical power losses:

$$u(t) = iR + N\frac{d\Psi(i,z)}{dt} \tag{2}$$

$$\begin{aligned}&\mathrm{curl}(\nu\,\mathrm{curl}\,\boldsymbol{A}) + \mathrm{curl}\left(\frac{\boldsymbol{M}}{\mu_1}\right) = \\ &= \boldsymbol{J} - \sigma\frac{\partial \boldsymbol{A}}{\partial t} + \sigma\boldsymbol{V}\times(\mathrm{curl}\,\boldsymbol{A})\end{aligned} \tag{3}$$

$$F_m(i,z) = m\frac{d^2 z}{dt^2} + B\frac{dz}{dt} + Kz + F_e \tag{4}$$

$$\rho C\frac{\partial T}{\partial t} - \nabla\cdot\left[k(T)\nabla T\right] = q^B \tag{5}$$

In the above equations $u(t)$, i and $\Psi(i, z)$, and z are the applied voltage, coil current, flux linkage with the coil, and the displacement of the valve plate respectively, R and N are the coil resistance and the number of turns in the coil, $\boldsymbol{J}$, $\boldsymbol{A}$, $\boldsymbol{V}$, $\boldsymbol{M}$ are the coil current density, magnetic vector potential, velocity, and magnetisation (for permanent magnets); m, B, K, F_m and F_e are the mass of the moving part, viscous damping coefficient, spring constant, magnetic force and the load force respectively; and T, and q^B are the temperature and the internal rate of heat generated per unit volume respectively. The material parameters ν, μ_1, σ, ρ, C and k denote the magnetic reluctivity (ν=1/μ, μ is the permeability), apparent permeability, electric conductivity, density, specific heat and the thermal conductivity respectively. In general, these equations are nonlinear and inseparable. The current produced by (2) creates the magnetic field given by (3) and produces the magnetic force which causes the displacement, speed and acceleration of the actuator obtained from (4). The current also generates the heat per unit volume and the resulting temperature distribution given by (5). There are two main approaches to the coupled solution of these equations: the direct coupled approach and the indirect coupled approach, neither of which alone is suitable to incorporate the whole array of factors which are expected to be encountered in the practical exploitation of high-speed actuators. These assume the necessity for qualitative and quantitative assessments of those factors that introduce nonlinearities in the system, in order to justify the use of one (or several) of these methods for modelling and simulation purposes. For example, if the motional and eddy-current effects are negligible, the winding inductance can be taken as constant and the decoupling of the equations would be adequate for dynamic analysis. On the other hand, the direct coupled approach would be more appropriate if the eddy-current effects are negligible, but saturation and motional nonlinearities are prominent. Thus a methodology which includes the provision for using both coupled or/and decoupled solutions of electromechanical equations by *a priori* qualitative and quantitative justification would be most appropriate. The methodologies for modelling and design of EM actuators are based upon the modelling and computation of the 2D/3D nonlinear magnetic field distribution using the numerical finite element (FE) technique [10]. This involves the steady-state and transient solutions of the nonlinear Poisson's equation for which there are no analytical solutions. These results are used for design optimisation and for investigating the effects of various geometric, material, EM and mechanical parameters on the output performance of EM actuators. The thermal modelling involves the development of 2D/3D thermal models and the FE solution of the steady-state and/or transient heat transfer equations given by (5) above. The heat sources needed for this are given by various losses mentioned above. The coupling of the magnetic field and the thermal equations (owing to the dependence of the power density on the magnetic vector potential and the temperature dependence of the magnetic permeability and electric conductivity) may be realised either by indirect coupling (in which the equations are solved separately and coupled by means of power density and an iterative process is used to compute the power density and the temperature distribution) or by direct coupling in which the equations are solved simultaneously.

3. REALISATION OF FINITE ELEMENT MODELS

As mentioned before, the capacitive sensor shown in Fig. 1 is used for angular position sensing in limited angle torque motor actuators used for frame scanning in infrared thermal imaging devices. It consists of a compact arrangement of four outer electrodes, one cylindrical centre electrode and two rotating electrodes. Potentials of the same magnitude but opposite polarities are given to the outer electrodes while the rotating electrodes are kept at zero potential. The rotating electrodes attached to the motor shaft rotate to change the effective overlap (and hence the field distribution) between the centre and the outer electrodes. This changes the capacitance between the electrodes and also the sensor output voltage V measured between the 'floating' centre electrode and the virtual earth. Here the main design variables are the geometric parameters of the electrodes and the dielectric material ε_7, and the output of interest is the magnitude and linearity of the voltage to angular position (V-θ) characteristic. The above modelling methodologies were used to carry out both 2D and 3D simulation studies to investigate the effects of various design parameters and the leakage flux on the performance of this large diameter to length ratio sensor. The modelling results were validated against experimental data obtained from industry.

A typical 3D FE model for the capacitive angular position sensor described above is shown in Fig. 3. Although in many cases it is usually sufficient to use 2D models for CAD and simulation studies of capacitive sensors, there are, however cases in which 3D models are required in order to take into account fully the complex nature of electric field distribution, especially in devices with complicated geometric configuration. This is especially true for cylindrical electrode configurations with small length-to-diameter ratio in which a detailed investigation of end effects may be required. In addition, sometimes 3D models are also required to justify, both qualitatively and quantitatively, the use of 2D models for simulation. Such a 3D FE model for the small length-to-diameter ratio angular position sensor is shown Fig. 3. It consists of about 75000 eight-noded hexahedral mesh elements made up of approximately the same number of nodes. For a relatively small size sensor (about 25 mm in length and 17 mm in diameter) this provides a modelling accuracy comparable to that of an accurate 2D model. Such 3D models were used to investigate the effects of leakage flux in the sensor [11]. In such cases, for boundary conditions either Dirichlet ($\Phi=\Phi_i$) or Neuman ($\partial\Phi/\partial n=k$) boundary condition is used. No boundary condition is imposed on any 'floating' potential regions (e.g. the centre electrode in Fig. 1). However special measures need to be taken to tackle these regions in FE modelling. For example, a novel technique was used for the first time to simulate the 'floating' potential centre electrode in the angular position sensor shown in Fig. 1 [6]. The idea is based on the fact that since it is a conductor there must not be any electric field inside the centre electrode ($\boldsymbol{E}=0$) and its surface must comprise an equipotential surface. This condition is satisfied by assigning an infinitely large permittivity value to all FE regions occupied by the centre electrode. This makes it unnecessary to model such floating electrodes as ‘holes’ simplifying FE discretisation and reducing modelling complexity.

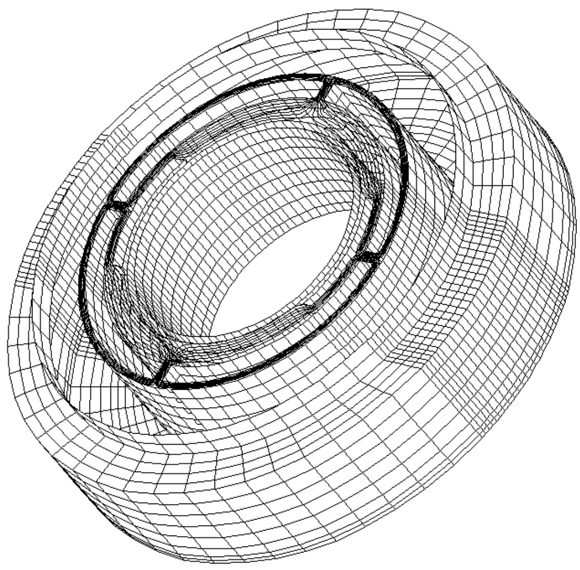

Fig. 4. Radial equipotential contours between the outer and centre electrodes in the capacitive angular position sensor.

Following the FE solution of the field equation (1), the capacitance C between the given electrodes is calculated either from the field energy E ($E=CV^2/2$) for a given potential difference V or from charge Q using $C=Q/V$. Here the charge Q is calculated by integrating the flux density vector $\boldsymbol{D}$ over the appropriate electrode surfaces [12] using the Gauss's law below:

$$Q = \oint_S D_n \, ds = \oint_S D\cos\theta \, ds = \oint_S \boldsymbol{D}\cdot \mathrm{n} \, ds = \oint_S \boldsymbol{D}\cdot \boldsymbol{ds} \qquad (6)$$

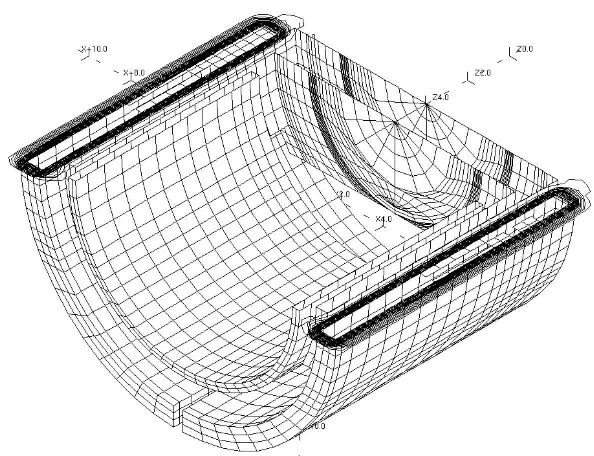

Fig. 5. Axial equipotential contours between the outer and centre electrodes in the capacitive angular position sensor.

It is clear that any errors in the capacitance calculation are dependent upon the errors associated with the field computation and the numerical integration technique used to calculate the charge Q from (6).

The limited-angle torque motor actuator shown in Fig. 2 consists of a solid iron armature surrounded by the solid stator made of the same ferromagnetic material. The armature is pressed on to the motor shaft made of nonmagnetic stainless steel. The stator consists of two segmented yokes (called the magnet yoke pair), which are fixed in position to the nonmagnetic end plates of the motor containing the bearing housings. The stator windings or excitation coils are wound around the yokes and are connected either in series or in parallel. The external diameter of the armature blades maintains a very small airgap of 0.05-0.07 mm between the armature and stator poles. When the stator windings are unexcited (I=0) the magnetic flux produced by the two permanent magnets resting on the stator segments produces a ‘detent torque’ which retains the armature at $\theta=0$ position shown in Fig. 2. Thus the motor has an inherent magnetic stiffness which, in many applications, provides for fail safe operation. The performance of this actuator is very much dependent upon the geometric and chosen material parameters for the stator yokes and armature, and the permanent magnets. The linearity and efficiency of its performance during an active scan period is affected by the degree of saturation of its magnetic circuit which, in turn depends on the materials used. Hence the importance of computer-based modelling and investigation, especially when presently there is a requirement for improved linearity of 0.1% during the active scan period and an increased scanning efficiency (active scan time/total scan time) of 90%. This is done by modelling and computation of 2D nonlinear magnetic field distributions in the actuator for various combination of its geometric and material parameters using the FEM.

For a given armature position θ the above equations (2-5) are solved either separately or simultaneously by the FEM in the problem domain defined by the FE model used. Usually, various modelling experiments are carried out to establish the appropriate number, size and distribution of elements that would ensure quick and efficient calculation of such important global parameters as torque, force, and other parameters of practical interest. Following the field computation, the torque/force is calculated by using one of the three methods based on one of the following three principles: Maxwell’s stress tensor, virtual work, and the magnetising current. In order to increase the overall modelling accuracy, about 27000 first-order triangular FE elements were used for modelling and analysis of the torque motor actuator shown in Fig. 2. The small size of the airgap (0.05-0.07 mm) in this actuator made the task of efficient and accurate discretisation in and around the airgap particularly challenging.

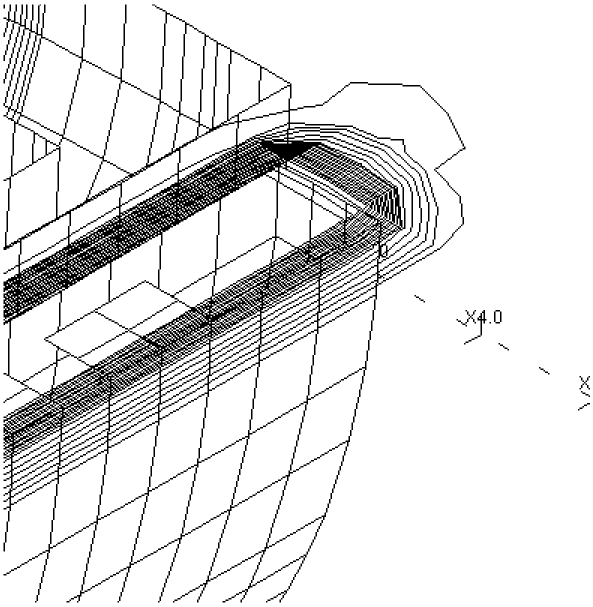

Fig. 6. Axial equipotential contours between the outer and centre electrodes showing the non-uniform end effects leading to nonlinear sensor performance.

4. SOME MODELLING RESULTS

Figs. 4-6 show the typical field distributions between the outer and centre electrodes for the capacitive angular position sensor shown in Fig. 1. The magnitude and linearity of its output voltage-position characteristic (V-θ) are very much dependent upon the uniformity of the field distribution and the position and size of the rotating electrodes [12]. For this reason it is especially important to characterise and quantify the end effects (shown in Figs. 5 and 6) by 3D modelling as they contribute to the non-uniformity of the above field distribution. Simulation results have shown that the V-θ characteristic is also critically dependent upon the dielectric property of the

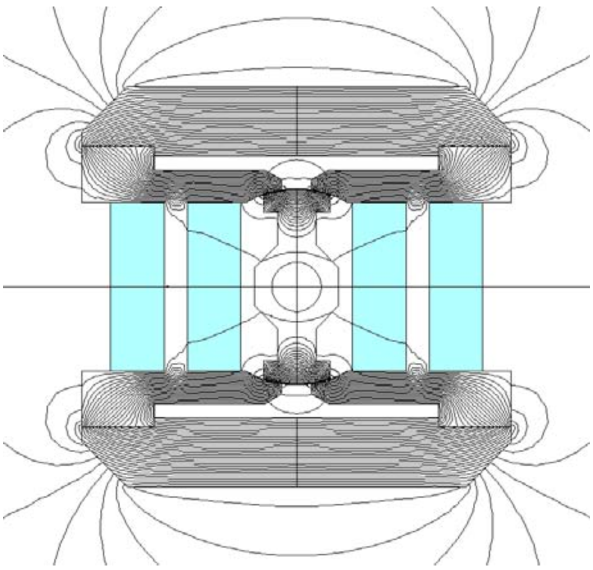

Fig. 7. Magnetic field distribution in the torque motor actuator (shown in Fig. 2) for armature position $\theta=0$.

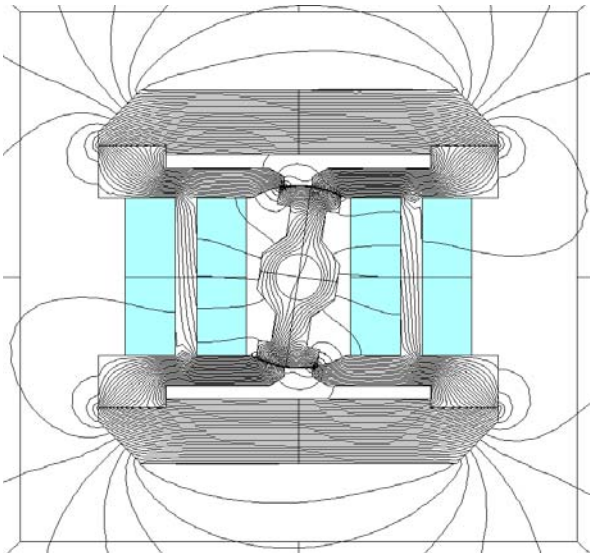

Fig. 8. Redistribution of magnetic field distribution in the torque motor actuator for armature position θ=–8°.

inner insulator ring (ε_7) between the centre electrode and its carrier (Fig. 1).

Some of the modelling results in terms of field distribution for the torque motor actuator are presented in Figs. 7 and 8. For zero excitation current (I=0), the magnitude and distribution of this field affect the magnetic stiffness which is a measure of its magnetic condition. Fig. 9 shows some of the torque-current characteristics for various armature positions θ. For a given armature position, the magnetic field distribution in the inner magnetic circuit depends not only on the degree of saturation but also on the direction of the excitation flux which has both magnetising and demagnetising effects in different parts of this circuit.

5. CONCLUSIONS

The examples presented in the paper show that modern computational tools provide powerful methods for modelling, analysis and design of sensors and actuators. With the availability of powerful computer hardware and software tools significant progress is being made in the application of advanced numerical techniques such as the FEM for modelling and analysis of sensors and actuators. These tools are an important addition to the methodology of measurement science.

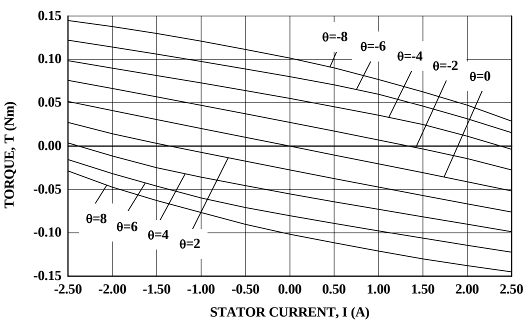

Fig. 9. Design curves obtained by finite element modelling for the torque motor actuator showing the variation of torque with excitation current for various armature positions θ.

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