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THE CALLIPER RULE AS A PARTICULAR CASE OF RESIDUE NUMBER SYSTEM ARITHMETIC

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Abstract – The caliper rule is generally considered as a particular tool, which renders a scale more readable. This paper shows that this rule can be viewed, from a theoretical point of view, as the way to realize a small “virtual unity”. This virtual unity can be used in determining the conversion between different units and can represent the basic unity of a particular measurement system, which gives the results of the measurements directly in RNS arithmetic.

Keywords Residue Number System, Scale, Caliper.

1. INTRODUCTION

This paper presents a measurement system that make use of the concept of “scale” recently discussed in the literature devoted to the fundamentals of measurements [1]. The measure is performed by a comparison of scales. This comparison determines a “virtual unit” that not exists physically but corresponds to the available accuracy of the system. This unit will be referred in the following as “basic unit” (BU). The system can be regarded as based on the Residue Number System arithmetic (RNS) and give the results of the measurements, as values in terms of the BU, represented by traditional or RNS arithmetic. Indeed it will be shown that the value and the residues of the value of an unknown quantity X can be determined by shifting one of the scales of a quantity X with respect to the others. A particular way in determining the conversion factors between different units, by using the described system will be illustrated. It will shown also that the caliper is a particular case of the general measurement system here presented. For the sake of clarity the system will be described as a length measurement system. However, the developed theory can be applied to the measure of any physical quantity.

2. THE PROPOSED SYSTEM

Suppose that to perform the length measurements we have available some physical segments having unknown lengths u_i , $i = 1, 2, ..., N$ and the property that a multiple or a sub multiple of u_i cannot “correspond”, in terms of our available accuracy, to u_j, for all the i, j. The expression "prime each other" cannot be used here because the values of the lengths are still unknown. By iteratively reporting these lengths over a straight line we can create a set of scales, of the type ϕ^1, following the notation given in [1], i.e. having the same origin but different units. Let n_i, $i = 1,..., N$ be the integer counters of this iterative procedure. Then, for some particular values $\tilde{n}_i$ of these counters, we have that

$$\left|\tilde{n}_i u_i - \tilde{n}_j u_j\right| = \min \ for\ all\ i, j \qquad (1)$$

Condition (1) is cyclic, so we choose the lower values for the $\tilde{n}_i$. Note that the assigned lengths u_i can also be incommensurable among each other. Our finite accuracy in comparing the lengths overcomes this eventuality. An illustration of this last assertion can be given by a simple example. We suppose that $N = 2$ and $u_1 = \pi$, $u_2 = e$ in terms of an arbitrary length unit (ALU). Then calculate the distances between the scale steps as a 100x100 matrix:

$$\Delta_{n_1,n_2} = \left|n_1 u_1 - n_2 u_2\right| = \left|n_1 \pi - n_2 e\right| \qquad (2)$$

A surface plot of these distances is given in the following figure 1.

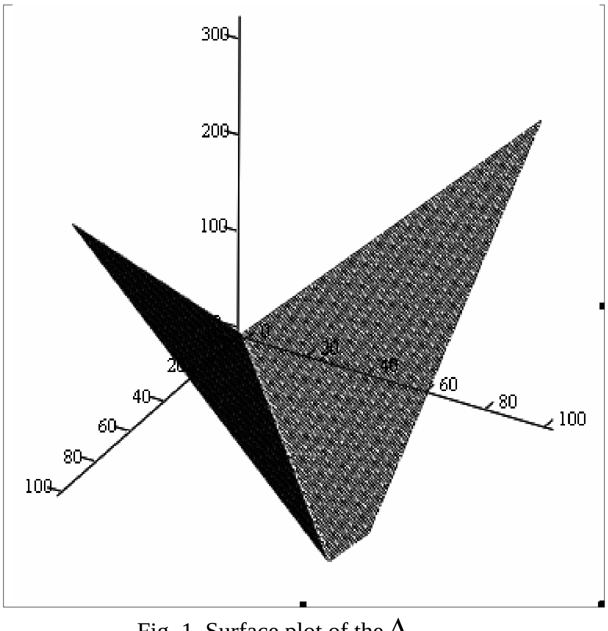

Fig. 1. Surface plot of the Δ_{n_1,n_2}

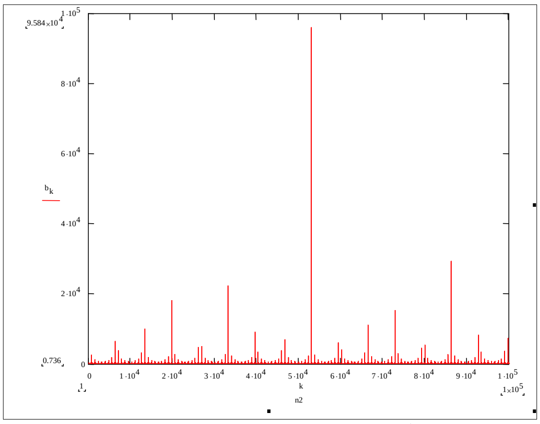

Fig. 2. Plot of the reciprocal of the distances Δ_{n_1,n_2} along the line given by expression (2)

The figure 1 shows that the lower values of the distances Δ_{n_1,n_2} occur along the line:

$$\bar{n}_1 = round\left(\frac{\pi}{e}\cdot n_2\right) \qquad (3)$$

Here "round" means "nearest integer". A calculation along this line of the values of

$$A_{n_2} = \frac{1}{|\bar{n}_1 e - n_2\pi|} \qquad (4)$$

gives the results shown in figure 2.

Figure 2 shows that, due to a finite accuracy in comparing the lengths, two theoretically different lengths, are assumed to be equal in practice. In the case given as example, if our accuracy is limited to $2\cdot 10^{-5}$ ALU, $\tilde{n}_1 = 53035$ and $\tilde{n}_2 = 61294$. Indeed $|53035\cdot\pi - 61294\cdot e| \approx 1.0434\cdot 10^{-5}$ ALU, corresponding to the central peak in figure 2.

On the basis of the given example, we can suppose that, in the general case, a set of values of the counters $\{\tilde{n}_i\}$ exists so that

$$\tilde{n}_i u_i \equiv \tilde{n}_j u_j \quad for\ all\ i, j \qquad (5)$$

where the symbol "≡" means "corresponds in terms of our accuracy" and will be replaced in the following by a simple "=". Recalling the property imposed to the u_i, condition (5) is verified when

$$\tilde{n}_i u_i = \prod_j^N u_j = M \quad , \ i = 1, 2, ..., N \qquad (6)$$

So we have that the following quantity:

$$\frac{n_i u_i}{\prod_j^N u_j} = 1 \quad , \quad i = 1, 2, ..., N \qquad (7)$$

can be assumed as the "Basic Unity" (BU) of this measurement system. However, this is a "virtual unity", very small with respect to the u_i, which not exists physically. Nevertheless, as happens in the caliper, the measurements can be performed with an accuracy related to this unity. This simply considering the n_i, given by direct observation, and some of their differences. From (4) it follows that:

$$\tilde{n}_i = \prod_{j\neq i}^N u_j \qquad (8)$$

Expression (8) means that also the u_i must be integer, as can be easily proved. Indeed the n_i are integer, so that the sentence is surely valid for $N = 2$. An induction procedure over N shows that if we have a system of order N and the related u_i, $i = 1, 2, ..., N$ that are integer, also a further $(N+1)^{th}$ length, inserted in the preceding system to obtain a system of order $N+1$, must be integer.

Now

$$\prod_i^N \tilde{n}_i = \prod_i^N \prod_{j\neq i}^N u_j = \left(\prod_j^N u_j\right)^{N-1} \qquad (9)$$

in such a way that

$$\prod_j^N u_j = M = \sqrt[N-1]{\prod_i^N \tilde{n}_i} \qquad (10)$$

and finally, from (2), the unknown values of the lengths u_i can be determined in terms of BU as:

$$u_i = \frac{M}{\tilde{n}_i}\ , \ i = 1, ..., N \qquad (11).$$

Let us consider the simple example of fig. 3. Here N=3. The u_i are firstly assigned as segments of unknown length. Let us consider the simple example of fig. 3. Here N=3. The u_i are firstly assigned as segments of unknown length.

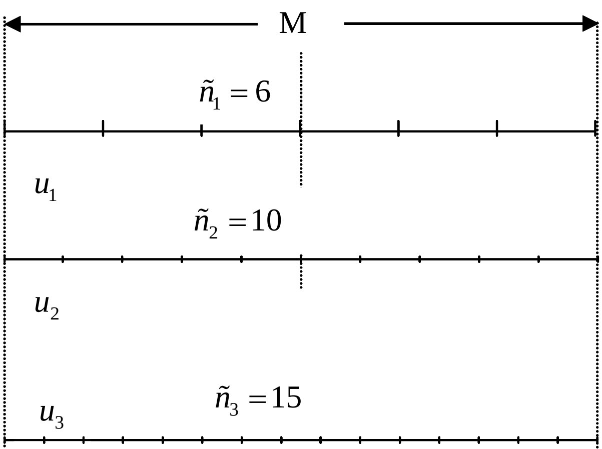

Fig. 3. Simple example of the length measurement system

We determine the values of the $\tilde{n}_i$ by inspection of the figure 3. Then the value of M results from (10) as $M = 30$ in terms of basic unity (7). The values of the u_i result from (11), in terms of BU, as:

$$u_1 = \frac{1}{6}\cdot 30 = 5 \;\; ; \;\; u_2 = \frac{1}{10}\cdot 30 = 3 \;\; ; \;\; u_3 = \frac{1}{15}\cdot 30 = 2 \;\; .$$

As we can see, all the quantities involved are represented by integer numbers. This is valid in the general case, as can be deduced from formulas (8-11).

3. MEASUREMENTS IN RNS

Since the values of the lengths u_i are integer, they can be interpreted as the "moduli" of a RNS System [2, 3]. In this system the residues with respect to the given moduli can represent each integer value measured (in terms of BU) $X \mid \;\; 0 \le X < M$.

Our particular system furnishes an analogue way for the calculation of the residues of an unknown quantity X (a length in our example) with respect to the given moduli. Let us consider the situation depicted in fig. 4: one of the scales of fig. 3 (the upper) is shifted with respect to the others of a length corresponding to the unknown length X . So the residues of this unknown length X can be obtained by inspection of the relative position of the first scale with respect to the others. This inspection determines firstly the steps of the scales that are more next among each other. Then, a simple difference between two lengths gives the residues. Let us explain in detail how the residues can be obtained.

We have that the steps of the "mobile" scale, the scale having step u_1 in our example, correspond to the lengths:

$$l_1(n_1) = X + n_1 u_1 \;\; , \;\; -\infty < n_1 < \infty \qquad (12)$$

and for all the "fixed" scales we have that:

$$l_j = n_j u_j \;\; , \;\; -\infty < n_j < \infty \qquad (13)$$

$j = 2, 3$ in our example.

We determine by inspection for each j-th scale the more next $l_1(n_1), l_j(n_j)$, $j = 2,\ 3$ in our example, namely $\overline{l}_1^{(j)}(\overline{n}_1^{(j)}) = \overline{l}_j(\overline{n}_j)$ (also in this case we have a cyclic possibility and choose the lowest values of $\overline{n}_1^{(j)}$ and $\overline{n}_j$). Now we remember that $l_1^{(j)}(\overline{n}_1^{(j)}) = X + \overline{n}_1^{(j)} u_1$ and $\overline{l}_j = \overline{n}_j u_j$, so that:

$$X = \overline{n}_j u_j - \overline{n}_1^{(j)} u_1 \;\; , \;\; j = 2,\ 3 \text{ in our example} \qquad (14)$$

Since for each of the j-th scale we can express the value of X in terms of the u_j and the related residues r_j , i.e.:

$$X = n_j^* u_j + r_j \qquad (15)$$

Using (14) and (15) we obtain:

$$r_j = (\overline{n}_j - n_j^*) \cdot u_j - \overline{n}_1^{(j)} u_1 \qquad (16)$$

The determination of the residue r_1 requires a change of the point of view from the left side to right side of fig. 4: we regard the first scale as fixed and the second as mobile.

So a correspondent of formula (16) can be evaluated as:

$$r_1 = (\tilde{n}_1 - \overline{n}_1^{(j)} + n_1^*) \cdot u_1 - (\tilde{n}_j - \overline{n}_j) \cdot u_j \;, \;\; j = 2 \; or \; 3 \qquad (17)$$

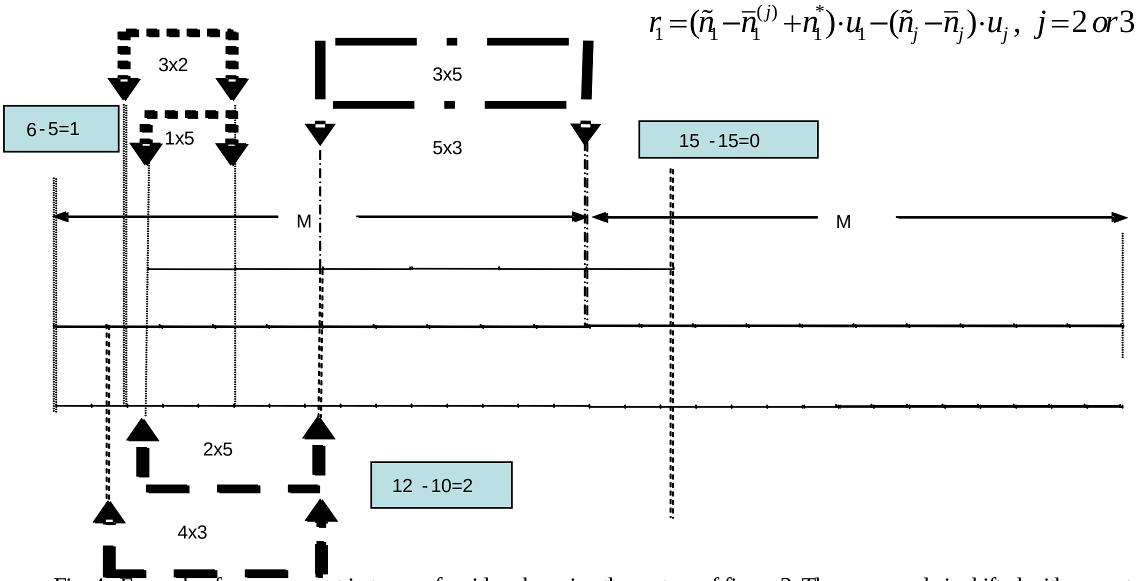

Fig. 4. Example of measurement in terms of residues by using the system of figure 3. The upper scale is shifted with respect to the others of a length $X = 5$ BU, to be measured in terms of residues; we obtain $r_2 = 2$ from formula (16), being $\overline{n}_2 = 5,\;\; n_2^* = 1,\;\; \overline{n}_1^{(2)} = 2$; again from formula (16), $r_3 = 1$, being $\overline{n}_3 = 5,\;\; n_3^* = 2,\;\; \overline{n}_1^{(3)} = 1$ and finally, from formula (17), $r_1 = 0$, being $\tilde{n}_1 = 6,\;\; \overline{n}_1 = 4,\;\; n_1^* = 1,\;\; \tilde{n}_2 = 10,\;\; \overline{n}_2 = 5$

As said above and explained in the caption of figure 4, the measurements system already presented gives the results of a measure of the quantity X in terms of residues with respect to the given lengths assumed as moduli of an RNS system. Is interesting to note that the same system can furnish the physical length X starting from its value represented by a set of residues with respect to the assumed moduli. To this purpose, we fix an origin and then shift each of the scales of a quantity equal to the related residue. The value of the distance between the origin and the first alignment of the steps of all the scales corresponds to the value of the physical quantity X .

.4. CONVERSION OF UNITS

A way to compare two different units, two length units as an example, consist in creating an auxiliary scale having a small unit, a small length in our example, and then compare the two units separately with this scale. The values obtained by this comparison can be used to determine the conversion factor between the two length units. If we want to determine the conversion factor between meter and yard, we create a scale having a suitable small unit (SU), e.g.$10^{-4} m$. Then compare the yard and the meter with this scale obtaining 9144 SU and 10000 SU respectively. Then a conversion factor between the two units can be calculated as 9144/10000 i.e. a yard corresponds to 0.9144 meters. However, the conversion factor depends on the chosen small unit of the auxiliary scale only in terms of accuracy. A different choice of the unit of the auxiliary scale gives a different approximation of the same value. Another way in determining the wanted conversion factor can be suggested from the measurement system described in the above sections 2 and 3. We create two scales, of the type ϕ^1, having units 1 meter and 1 yard respectively and the same origin. We count the steps of the two scales until we reach another alignment. If we suppose an accuracy of the order of $10^{-4} m$ in the alignment and ideal scales having equal steps, we find another alignment after 1250 steps in the "yard" scale and 1143 steps in the "meter" scale. Then, using the relation (5), we determine the conversion factor directly as 1143/1250=0.9144. Obviously, this procedure is unpractical and has in this case only conceptual value. However it can be very useful in other measurements, i.e. in determining the period or the frequency of one or more periodic signals using a sinusoidal reference signal or in determining the period of revolution of a satellite with reference to another one. But this procedure is often already in use without any reference to the scales. As an example, in a hardware store the shop assistant measures in the described way the mass ratio among the bolts and the related nuts, using a two pans balance, to avoid tedious counts. He poses the bolts in the right pan and the nuts in left pan until the pans are aligned. The ratio between the number of the bolts and the nuts respectively gives the ratio between the related masses.

5. CALLIPER RULE

We recall the caliper properties related to the formulas already given for the general case of N moduli. In the caliper, the two moduli are determined in such a way that the same interval *M* is divided by $\tilde{n}_1$ in the fix scale and by $\tilde{n}+1$ in the mobile one. The particular choice of the moduli allows that $\overline{n}_2 = \overline{n}_1 = \overline{n}$, i.e. that the number of steps involved in both the two scales is the same. From formula (14), we have that:

$$X = \overline{n} \cdot \frac{M}{\tilde{n}} - \overline{n} \cdot \frac{M}{\tilde{n}+1} = \frac{M}{\tilde{n}} \cdot \frac{1}{\tilde{n}+1} \cdot \overline{n} \qquad (18)$$

In the caliper, typically $\tilde{n} = 19$ so that $M = 380\ BU$; M is chosen as $19\ mm$ so that the BU results $1/20\ mm$.

6. CONCLUSION

The concept of scale was used to establish a connection between the Residue Number System and a measurement procedure. It was shown that, by means of comparisons between scales having different unit steps, a measurement can be performed giving the result in terms of traditional and RNS arithmetic. The proposed measurement system performs also the conversion from an RNS representation and the physical dimension of the measured quantity. The proposed system was described as a length measurement system but it can construct and applied to the measurements of any physical quantity. The use of the system as tool for conversion of units was illustrated. It was shown also that the Caliper Rule can be regarded as a particular case of the proposed measurement system.

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