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THEORY OF A CORIOLIS MASS FLOWMETER INSERTION PROBE

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A novel Coriolis mass flowmeter insertion probe concept is described suitable for the measurement of the mass flow of liquids flowing in pipes. It is generally similar to that of the turbine meter insertion probe for volume flow measurement except that the turbine is replaced by a small vibrating aerofoil of elliptical cross-section. The weight vector theory of Coriolis mass flowmeters is applied to predict the approximate sensitivity of the probe. The sensitivity so predicted can be comparable to that of commercial (vibrating tube) Coriolis mass flowmeters so that the signal processing techniques developed for these could be directly used in conjunction with the proposed insertion meter. The dependence of sensitivity on Reynolds number and on liquid density is discussed and a way of removing dependence on liquid density is described.

Introduction

The theory of a novel Coriolis mass flowmeter insertion probe is presented. This insertion probe applies the principle of Coriolis mass flow measurement to provide a measurement of the local value of fluid density times flow velocity (i.e. the quantity ρv). By scanning the value of this quantity over a pipe diameter the total *mass* flow rate of the fluid can be measured. This is in contrast to other insertion probes [1] using turbine, differential pressure, electromagnetic or ultrasonic principles which can measure the total *volume* flow rate.

The novel insertion probe resembles an insertion probe employing for example a small turbine flowmeter near the end of a rod, but has in place of the turbine, a small vibrating aerofoil of elliptic cross-section over which flow passes (Fig 1). The aerofoil is made to vibrate in a direction perpendicular both to the flow direction and its own axis. The effect of flow over the aerofoil is to produce a Coriolis reaction couple tending to cause a secondary (rotary) vibration of the cylinder a bout its axis. The secondary vibration gives rise to a small phase difference or time delay between the motion of the leading and trailing edges of the aerofoil. This time delay is proportional to ρv and can be measured by phase comparison of the signals from transducers sensing the motion of parts of the aerofoil near the leading and trailing edges.

The weight vector for a Coriolis mass flowmeter of a configuration similar to that considered here has been calculated analytically [2] and described in references [3,4]. This gives the weight that the flowmeter attaches to fluid velocity in any small volume of space near the elliptical aerofoil. In this paper we explain (as shown in [2]) how the weight vector can be combined with potential flow around the aerofoil to calculate the sensitivity of the probe. We show that the demands placed on flowmeter electronics to measure the time differences involved need be no greater than in conventional (vibrating tube) Coriolis mass flowmeters.

In general the sensitivity of the probe (i.e. the constant of proportionality between the time difference between signals from the transducers and the value of ρv) depends on fluid density ρ and to some extent on Reynolds number. These dependences are discussed in the paper and we explain how density dependence can be removed by designing the mass distribution within the probe so that a certain relation is satisfied between the radius of gyration of the aerofoil and the major and minor axes of its elliptical cross-section.

Insertion probe design concept

Various alternative designs of Coriolis mass flowmeter insertion probes are of course possible, but for the purpose of the present study it will be necessary to fix our ideas concerning the general electromechanical construction.

Thus we suppose that the aerofoil is a stainless steel shell housing a pair of accelerometers as the sensing transducers and a permanent magnet to make possible the electromagnetic driving of the aerofoil vibration by excitation of a coil housed in the probe body just outside the channel containing the aerofoil (Fig. 2). The aerofoil is supported by 4 ribbons of stainless steel rigidly embedded into the plane stainless steel ends of the aerofoil and into the back of recesses in opposite sides of the aerofoil channel (Fig. 3).The ribbons act as springs, allowing vibration of the aerofoil in the y-direction (the driven motion) and rotation of the aerofoil about the z-axis (Coriolis reaction).
In operation the coil is energised by an ac current to cause the aerofoil to vibrate (in resonance) as a rigid body with an amplitude of vibration (of perhaps 70microns) small compared to the aerofoil dimensions and the boundary layer thickness over the aerofoil. At the same time the ac signals from the accelerometers are compared to determine the phase difference $\Delta\phi$ or the time difference Δt between the motions of two parts of the aerofoil situated near the leading and trailing edges. A linear relation exists between Δt and the mass flow flux density ρv in the aerofoil channel. Thus

$$\frac{\Delta t}{\rho v} = S \qquad (1)$$

where S is a constant, 'the sensitivity of the probe'. As we will see S is not an absolute constant. It can depend quite strongly on liquid density unless the aerofoil mass is properly distributed. It is also somewhat dependent on Reynolds number which we define as

$$\mathrm{Re} = \frac{bv}{\nu} \qquad (2)$$

where b is the semi-minor axis of the elliptical cross-section of the aerofoil , v the incident flow velocity (flat profile assumed at the inlet to the aerofoil channel) and ν is the kinematic viscosity of the liquid. Since the aerofoil is vibrating at resonance (e.g. by means of feedback of a accelerometer signal) the frequency f_0 of vibration is (neglecting damping)

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{\bar{k}}{M + M_e}} \qquad (3)$$

where $\bar{k}$ is the spring constant for displacement of the aerofoil in the y-direction, M is the mass of the aerofoil and all within it and M_e is the additional effective mass of the liquid.

<u>Calculation of frequency of operation</u>

We obtain a formula for the frequency of operation using (3) with the following expressions for $\bar{k}$, M and M_e.

$$\bar{k} = 4E\frac{fg^3}{l^3} \qquad (4)$$

$$M = \pi\rho_s(abh - (a-\delta)(b-\delta)(h-2\delta)) \qquad (5)$$

$$M_e = \pi\rho a^2 h \qquad (6)$$

Equation (4) is derived by treating each ribbon in Fig.3 as a beam (length l) clamped at its ends and applying simple beam theory.
Equation (5) assumes the aerofoil shell has the form of a hollowed out solid with elliptical cross-sectional boundaries inside and outside and with flat end plates of thickness δ (the same as the thickness of the shell in the direction of the minor or major axis). The masses of the transducers and magnet are neglected in (5).
Equation (6) can be inferred from the expression for the kinetic energy of the liquid around a moving elliptical cylinder (see for example p129 of [5]).
With the following particular values of constants, i.e.
$a = 5$ mm, $b = 1$ mm, $\delta = 0.302$ mm, $h = 20$ mm

$\rho_s = 7.8 \times 10^3\ \text{kg/m}^3,\ \rho = 10^3\ \text{kg/m}^3$

$f = 1\ \text{mm},\ g = 0.1\ \text{mm},\ l = 2\ \text{mm}$

$E = 21.55 \times 10^{10}\ \text{N/m}^2$ ... (7)

we have

$\bar{k} = 1.077 \times 10^5\ \text{N/m}$

$M = 0.892\ \text{gms}$

$M_e = 1.57\ \text{gms}$

$f_0 = 1052\text{Hz}$ ... (8)

<u>Derivation of the weight vector</u>

The weight vector $\boldsymbol{W}_f$ relates the small phase difference $\Delta\phi$ between the sensor signals to the flow velocity distribution $\boldsymbol{v}$ (time averaged in the case of turbulent flow). Thus

$$\Delta\phi = \int \boldsymbol{v}.\boldsymbol{W}_\phi dV \quad (9)$$

the integral being conducted over the entire volume occupied by the liquid.
The weight vector itself is given by

$$\boldsymbol{W}_\phi = 2\rho d\, \text{Im}((\boldsymbol{u}^{(1)}.\nabla)\boldsymbol{u}^{(2)} - (\boldsymbol{u}^{(2)}.\nabla)\boldsymbol{u}^{(1)}) \quad (10)$$

where $\boldsymbol{u}^{(1)}$ is the (complex) amplitude of the vibrational velocity field set up in the liquid by an aerofoil vibration in the y-direction with unit velocity amplitude and $\boldsymbol{u}^{(2)}$ is the (complex) amplitude of the vibrational velocity field set up in the liquid due to a couple of unit amplitude in the –ve z-direction causing rotational vibration of the aerofoil about its axis (the z-axis in Fig. 1). (The actual velocity fields are understood to be the real and imaginary parts of $\boldsymbol{u}^{(1)}e^{i\omega t}$ and $\boldsymbol{u}^{(2)}e^{i\omega t}$ where $\omega = 2\pi f_0$.)
A derivation of (10) is given in [2]. It can also be derived more directly from equations (15)(or (21)) and (19) of [3] by taking points P, P', C of Fig.3 of [3] to be respectively at the centres of the left and right accelerometers and the magnet in Fig. 2 of the present paper and noting that the couple produced by equal and opposite unit forces at P and P' is equal to $2d$ in the –ve z-direction. Note that (9) and (10) assume that the effect on $\Delta\phi$ of liquid viscosity in the vibrational flows $\boldsymbol{u}^{(1)}$ and $\boldsymbol{u}^{(2)}$ is negligible. As explained in [2] and [4], a (usually small) correction to account for viscosity in the vibrational flows can be accommodated in terms of a surface vorticity weight vector but this correction is not carried out in the present paper.
The aerofoil channel (Fig. 1) is assumed wide enough in the y-direction and long enough in the x-direction for us to calculate $\boldsymbol{W}_f$ as if the aerofoil was situated in a 2-D infinite ocean. Expressions for $\boldsymbol{u}^{(1)}$, $\boldsymbol{u}^{(2)}$ and $\boldsymbol{W}_f$ are then most easily obtained using elliptic coordinates ξ and η (Fig. 4). The relationship to the Cartesian coordinates x and y is

$$x = c\cosh\xi\cos\eta$$

$$y = c\sinh\xi\sin\eta$$

The unit vectors are

$$\boldsymbol{e}_\xi = \frac{1}{Q}c\sinh\xi\cos\eta\boldsymbol{i} + \frac{1}{Q}c\cosh\xi\sin\eta\boldsymbol{j}$$

$$\boldsymbol{e}_\eta = -\frac{1}{Q}c\cosh\xi\sin\eta\boldsymbol{i} + \frac{1}{Q}c\sinh\xi\cos\eta\boldsymbol{j}$$

$$Q = c\sqrt{\sinh^2\xi + \sin^2\eta}$$

and the del operator is

$$\nabla = \boldsymbol{e}_\xi \frac{1}{Q}\frac{\partial}{\partial\xi} + \boldsymbol{e}_\eta \frac{1}{Q}\frac{\partial}{\partial\eta}.$$

Expressions for $\boldsymbol{u}^{(1)}$ and $\boldsymbol{u}^{(2)}$ (p.128,135 of [5]) are

$$\boldsymbol{u}^{(1)} = -\nabla\phi^{(1)}, \quad \boldsymbol{u}^{(2)} = -\nabla\phi^{(2)},$$

$$\phi^{(1)} = a\sqrt{\frac{a+b}{a-b}}e^{-\xi}\sin\eta,$$

$$\phi^{(2)} = \frac{1}{4}\Omega(a+b)^2 e^{-2\xi}\sin 2\eta$$

where Ω is the angular velocity amplitude of the aerofoil (about the z-axis) due to the couple of unit amplitude. More explicitly

$$\boldsymbol{u}^{(1)} = a\sqrt{\frac{a+b}{a-b}}\frac{e^{-\xi}}{Q}(\boldsymbol{e}_\xi \sin\eta - \boldsymbol{e}_\eta \cos\eta),$$

$$\boldsymbol{u}^{(2)} = \frac{1}{2}\frac{\Omega(a+b)^2}{Q}e^{-2\xi}(\boldsymbol{e}_\xi \sin 2\eta - \boldsymbol{e}_\eta \cos 2\eta) \qquad (11)$$

Since $\nabla.\boldsymbol{u} = 0$ we have

$$(\boldsymbol{u}^{(1)}.\nabla)\boldsymbol{u}^{(2)} - (\boldsymbol{u}^{(2)}.\nabla)\boldsymbol{u}^{(1)} = -\nabla\times(\boldsymbol{u}^{(1)}\times\boldsymbol{u}^{(2)})$$

and by (11)

$$\boldsymbol{u}^{(1)}\times\boldsymbol{u}^{(2)} = \psi(\xi,\eta)\boldsymbol{e}_z,$$

$$\psi = \frac{\Omega}{2Q^2}a\frac{(a+b)^{5/2}}{(a-b)^{1/2}}e^{-3\xi}\sin\eta$$

so by (10)

$$\boldsymbol{W}_\phi = -\rho d\,\mathrm{Im}(\Omega)\boldsymbol{Z} \qquad (12)$$

with

$$\boldsymbol{Z} = (\sinh^2\xi + \sin^2\eta)^{-3/2} a\frac{a+b}{(a-b)^2}e^{-3\xi}\left[\boldsymbol{e}_\xi \cos\eta(1 - \frac{2\sin^2\eta}{\sinh^2\xi + \sin^2\eta}) + \boldsymbol{e}_\eta \sin\eta(3 + \frac{2\sinh\xi\cosh\eta}{\sinh^2\xi + \sin^2\eta})\right] \qquad (13)$$

The magnitude of $\boldsymbol{Z}$ falls very rapidly on moving away from the vicinity of the aerofoil. (See [2] and [3] for graphical portrayal of the weight vector.)

<u>General expression for sensitivity and removal of its dependence on liquid density</u>

Combining (12) and (9) we have

$$\Delta\phi = -\rho h d\,\mathrm{Im}(\Omega)Z^* \qquad (14)$$

with

$$Z^* = \int \boldsymbol{v}.\boldsymbol{Z}dA \qquad (15)$$

where the integration is conducted over the area of the xy-plane outside the cross-section of the aerofoil.

Assuming 2-D flow over the aerofoil it is clear that Z^* may be written as

$$Z^* = vF(\mathrm{Re}, a/b)b^2 \qquad (16)$$

where v is the same velocity as in (2) and F is some function of two variables reflecting the dependence of $\Delta\phi$ on flow velocity distribution over the aerofoil and on the aerofoil aspect ratio. (Note that we neglect the effect of boundary layers on the aerofoil channel walls.)

We also have

$$\mathrm{Im}(\Omega) = \omega^{-1}\left[(I + I_e) - (M + M_e)d_s^{\,2}\right]^{-1} \qquad (17)$$

which easily follows from the equation of motion

$$-k\theta - 1 = (I + I_e)\ddot{\theta} \qquad (18)$$

for the angle θ of twist of the aerofoil about the $+$ve z-direction under the action of a unit couple in the $-$ve z-direction. In (18) k is spring constant for twisting related to $\bar{k}$ by

$$k = \bar{k} d_s^{\,2}, \qquad (19)$$

I is the moment of inertia of the aerofoil about the z-axis and I_e is the additional effective moment of inertia of the liquid given by (p.88 of [6])

$$I_e = \frac{1}{8}\pi\rho c^4 h\,. \qquad (20)$$

Substituting $\theta = \Omega/i\omega$ and $\ddot{\theta} = i\omega\Omega$ into (18) and using (3) for $\bar{k}$ in (19) we get the result (17).
Putting $\Delta\phi = \omega\Delta t$ and collecting together the results (14), (16), (17), (4), (3) we arrive at the following general expression for sensitivity for the present configuration.

$$S = \frac{\Delta t}{\rho v} = FR\frac{l^3 h d}{4Efg^3} \qquad (21)$$

where F is the dimensionless function in (16), i.e.

$$F = \frac{Z^*}{vb^2} = \frac{1}{vb^2}\int \boldsymbol{v}.\boldsymbol{Z}dA \qquad (22)$$

and R is given by the dimensionless expression

$$R = \frac{b^2}{d_s^{\,2} - \dfrac{I + I_e}{M + M_e}}\,. \qquad (23)$$

Now R is generally a function of liquid density ρ because I_e and M_e are both proportional to ρ. On the other hand F is only weekly dependent on ρ through its dependence on Reynolds number. We can remove the main dependence of sensitivity S on ρ by making R independent of ρ.
By (23) R is independent of ρ when

$$\frac{I}{M} = \frac{I_e}{M_e} \qquad (24)$$

or, by (6) and (20), when the radius of gyration r_g of the aerofoil (and all its contents) satisfies

$$r_g = \frac{I}{M} = \frac{1}{8}\frac{(a^2 - b^2)^2}{a^2}\,. \qquad (25)$$

Condition (25) is satisfied (see p.52 of [2]) when the proportions of the hollow aerofoil satisfy

$$\frac{a}{b} = 5, \quad \frac{\delta}{b} = 0.302, \quad \frac{h}{b} = 30 \qquad (26)$$

provided the mass of the contents of the aerofoil can be neglected.
When condition (24) applies we have from (6), (20) and (23) that

$$R = \frac{a^2 b^2}{a^2 d_s^{\,2} - \dfrac{1}{8}(a^2 - b^2)^2} \qquad (27)$$

<u>Sensitivity to laminar flow and potential flow</u>

Values of F as given by (22) have been computed [2] for steady 2-D laminar flow over the aerofoil using FIDAP CFD code. This was carried out for an aerofoil of aspect ratio $a/b = 5$ in the Reynolds number range 100 to 2000. The computed values of F range from 107.70 (for $\mathrm{Re} = 100$) to 132.03 (for $\mathrm{Re} = 2000$) and are plotted in Fig. 5. (See [2] for a description of the mesh used and for computed flow patterns which generally show recirculation in a wake downstream of the aerofoil.) The computation assumes steady laminar flow and therefore cannot include effects of vortex shedding (if any is present).

The value of F for potential flow can be found analytically as follows. For potential flow over the elliptical aerofoil we have (p.131 of [5])

$$\frac{1}{v}\boldsymbol{v} = -\nabla\phi,$$
$$\phi = b\sqrt{\frac{a+b}{a-b}}e^{-\xi}\cos\eta + \sqrt{a^2-b^2}\cosh\xi\cos\eta. \qquad (28)$$

Since $\nabla.\boldsymbol{W}_\phi$ and $\nabla.\boldsymbol{Z}$ are zero we have the transformation

$$\int \boldsymbol{v}.\boldsymbol{Z}dA = -\int \nabla\phi.\boldsymbol{Z}dA = -\int \nabla.(\phi\boldsymbol{Z})dA = -\int_L (\phi\boldsymbol{Z})_n dL = \int_{-\pi}^{\pi} (\phi(\boldsymbol{Z})_\xi)_{\xi=\xi_0} Q d\eta \qquad (29)$$

where L is the boundary of the elliptical cross-section of the aerofoil on

$$\xi = \xi_0 = \frac{1}{2}\ln\frac{a+b}{a-b}. \qquad (30)$$

Combining (28), (13), (29) and (22) we obtain

$$F = 4\frac{a}{a+b}\frac{a^2+2ab-b^2}{b^2}\int_0^{\pi/2}\frac{\sinh^2\xi_0 - \sin^2\eta}{(\sinh^2\xi_0 + \sin^2\eta)^2}\cos^2\eta d\eta$$

The integral can be evaluated analytically and becomes

$$\frac{\pi}{2}(1-\tanh\xi_0)$$

and on account of (30) $\tanh\xi_0 = b/a$. Hence for potential flow we have

$$F = 2\pi\frac{a}{a+b}\frac{a^2+2ab-b^2}{b^2}(1-\frac{b}{a}). \qquad (31)$$

With $a/b = 5$ this gives $F = 142.42$. This is the value towards which the laminar value of F approaches as Re becomes large (see Fig. 5). We might expect this to be approximately the case even though a wake may persist in the laminar flow as $\mathrm{Re} \to \infty$ whereas no wake is allowed for in the above potential flow.

<u>Sensitivity of a particular probe geometry.</u>

With the proportions

$$\frac{a}{b} = 5, \quad \frac{h}{b} = 20, \quad \frac{d}{a} = \frac{2}{3}, \quad \frac{d_s}{a} = \frac{2}{3} \qquad (32)$$

we have from (27) (which assumes the density independent condition (25) is satisfied) that $R = 0.121$, so with potential flow the sensitivity is, by (21),

$$S = \frac{\Delta t}{\rho v} = 288\frac{l^3 b^2}{Efg^3}. \qquad (33)$$

With $b = 1$ mm, $f = 1$ mm, $g = 0.1$ mm, $l = 2$ mm, and $E = 21.55\times10^{10}$ N/m^2 (as in (7)) this gives

$$S = \frac{\Delta t}{\rho v} = 1.06\times10^{-8} \ \mathrm{m^2 s^2/kg},$$

so, for example, with water flow at 1 m/s,
$\Delta t = 10.6\mu s$.

At a frequency of $f_0 = 1052$ Hz (as calculated in (8)) this is a phase difference of

$\Delta\phi = 0.070\,\mathrm{rad} = 4.01^\circ$.

If the ratios a/b, h/b and d_s/a are kept constant and condition (25) is maintained equation (33) gives the scaling of sensitivity with aerofoil size (represented by b) and ribbon dimensions (l, f and g).

<u>Conclusion</u>

The theoretical study in this paper supports the feasibility of a certain kind of insertion probe for mass flow measurement which uses the Coriolis effect on a vibrating aerofoil. The local value (ρv) of mass flux density in the pipe flow is measured by the probe. (In all our analysis we have assumed that the value of ρv in the aerofoil channel is equal to the value of ρv at the same location in the pipe in the absence of the probe. If this is not so, because of Reynolds number effects in the aerofoil channel inlet or because of proximity to pipe walls, then of course an appropriate correction factor is needed in equation (21) for probe sensitivity S.)

The sensitivity S and frequency f_0 of operation of the device have been estimated in a particular case in which the insertion probe rod would be about 35mm diameter and would house a vibrating aerofoil of elliptical cross-section 20mm long. Calculated values of frequency of operation and sensitivity for this case are similar to the frequency of operation and sensitivity of certain commercial (vibrating tube) Coriolis mass flowmeters. Therefore the signal processing from these commercial meters could be more or less used directly in connection with the insertion probe proposed in this paper.

We have shown how, by adjusting the distribution of mass within the aerofoil so that the radius of gyration of the aerofoil about its central axis bears a certain relation to the major and minor diameters of its elliptical cross-section, the probe sensitivity (to mass flow) can be rendered independent of the density of the liquid.

In addition to measuring the mass flow, the proposed insertion probe could provide a measurement of the liquid density ρ (including a profile of ρ versus depth of immersion, in the case of spatially varying density). This could be achieved, as in conventional Coriolis meters, by monitoring the natural frequency f_0 of the vibration set up during operation and employing equation (3) to convert this to a measurement of ρ.

The formulae provided in the paper can be used to consider the effect on f_0 and S of scaling of the size of the probe to smaller or larger dimensions. Conditions for constancy of f_0 and S over a size range could then be established and doubtless achieved by suitable choice of ribbon dimensions and mass of the aerofoil.

Laminar CFD studies reported in the paper show an increase of probe sensitivity with Reynolds number. This is at about 16% per decade of Reynolds number in the Reynolds number range 100 to 1000 and is due mainly to the variation in the boundary layer thickness over the aerofoil. Other Reynolds number effects may be associated with the inlet flow into the aerofoil channel or with proximity of the probe end to the pipe wall.

Compensation for Reynolds number effects would require knowledge of the liquid viscosity. This might be achieved by monitoring of the current needed in the electromagnet to achieve a certain amplitude of vibration (as measured by the sensors). Because of damping effects in the viscous layer of the vibrational flow around the aerofoil, this current would be related to the liquid viscosity in a manner that could be quantified theoretically. The ability to measure the viscosity of the liquid (including the determination of a profile of viscosity versus depth of immersion, in the case of spatially varying viscosity) would be an additional advantage of the proposed insertion probe.

References

[1] Flow Measurement Handbook, R. C. Baker, CUP, 2000

[2] The theory of compact Coriolis mass flowmeters (PhD Thesis), L. A. Hendry, Cranfield University, October 1997

[3] The weight vector theory of Coriolis mass flowmeters, J. Hemp, FMI 5, 4, 247-253, 1994

[4] The weight vector theory of Coriolis mass flowmeters -Part 2. Boundary source of secondary vibration, J. Hemp and L. A. Hendry, FMI 6, 4, 259-264, 1995

[5] Fluid Dynamics, V. L. Streeter, McGraw-Hill, 1948

[6] Hydrodynamics, H. Lamb, 6th Ed., CUP, 1975

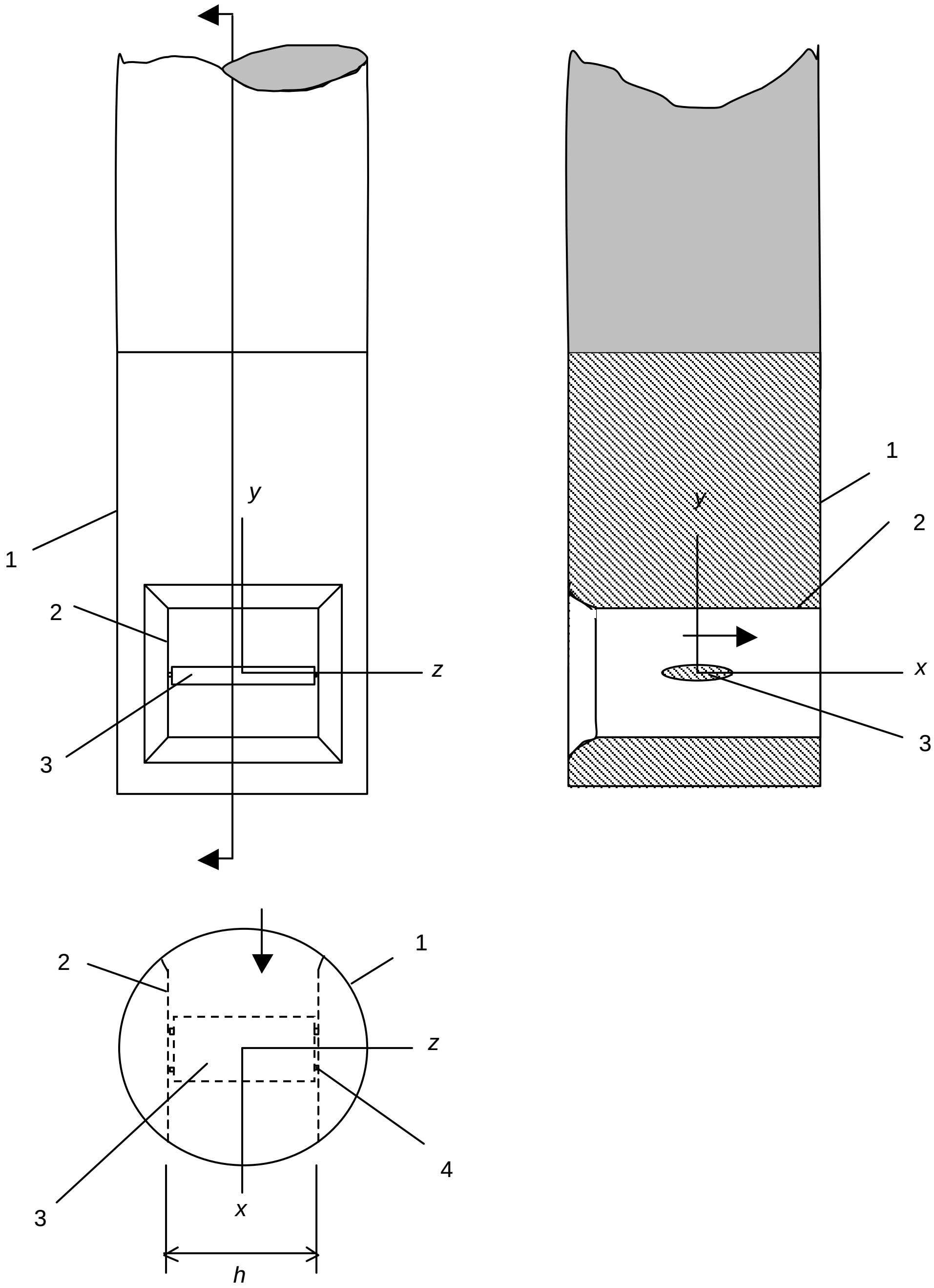

Fig 1 Three views of the Coriolis mass flowmeter probe
1. probe rod, 2. aerofoil channel, 3. aerofoil, 4. spring supports

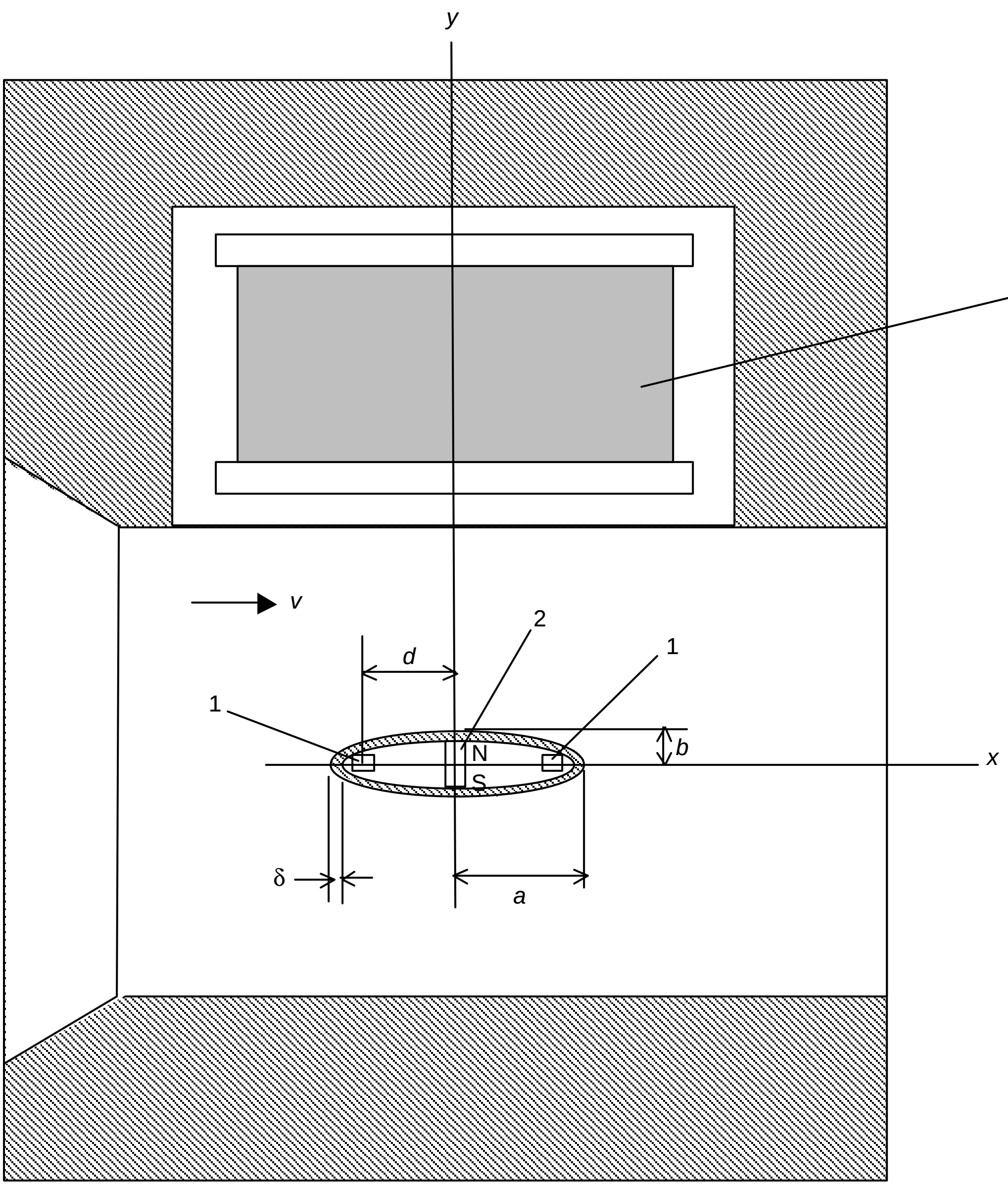

Fig 2 Sectional view of the Coriolis mass flowmeter probe showing transducers
1. accelerometer, 2. permanent magnet, 3. coil.

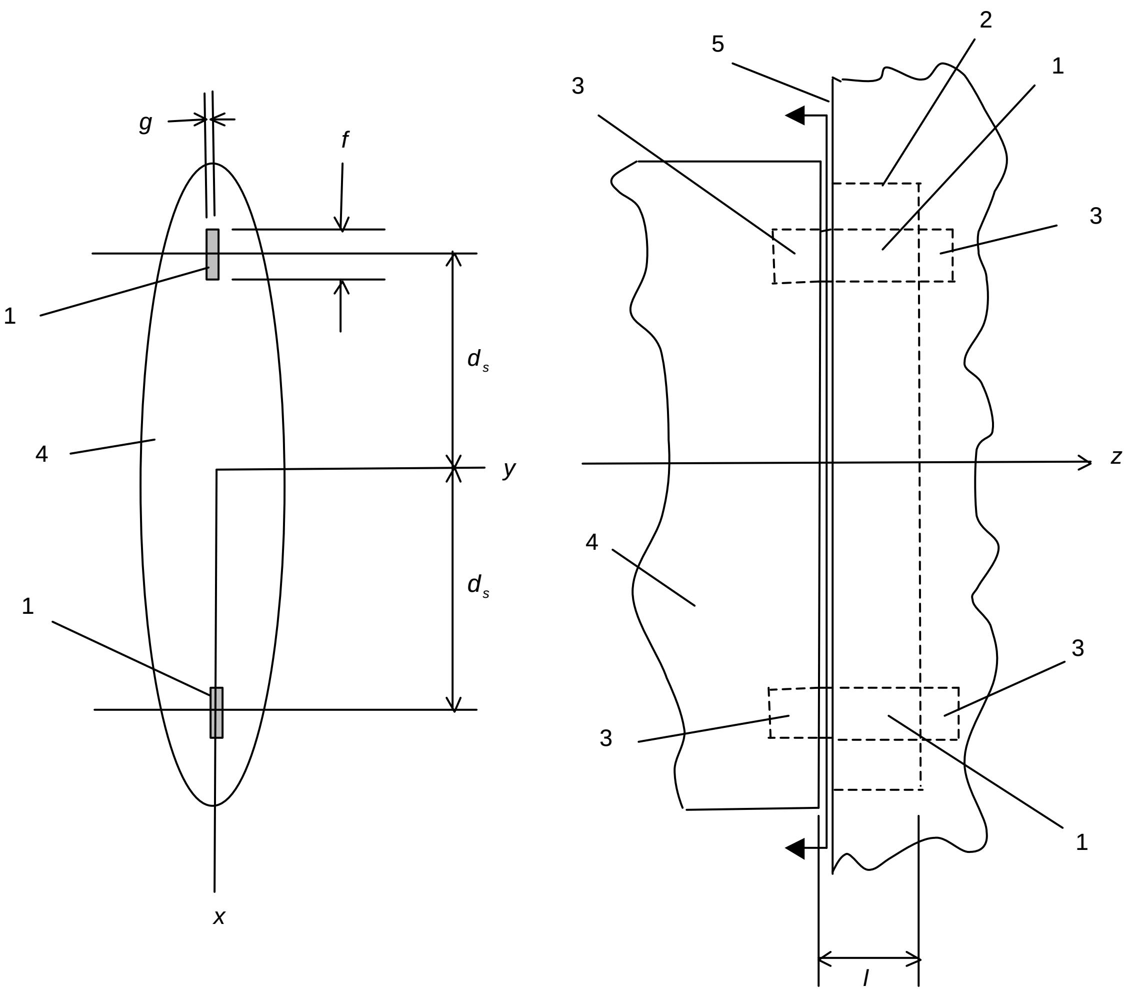

Fig 3 View of one end of the aerofoil showing ribbons used to support it.
1. ribbon, 2. recess in channel wall, 3. embedded end of ribbon,
4. aerofoil, 5. aerofoil channel wall.

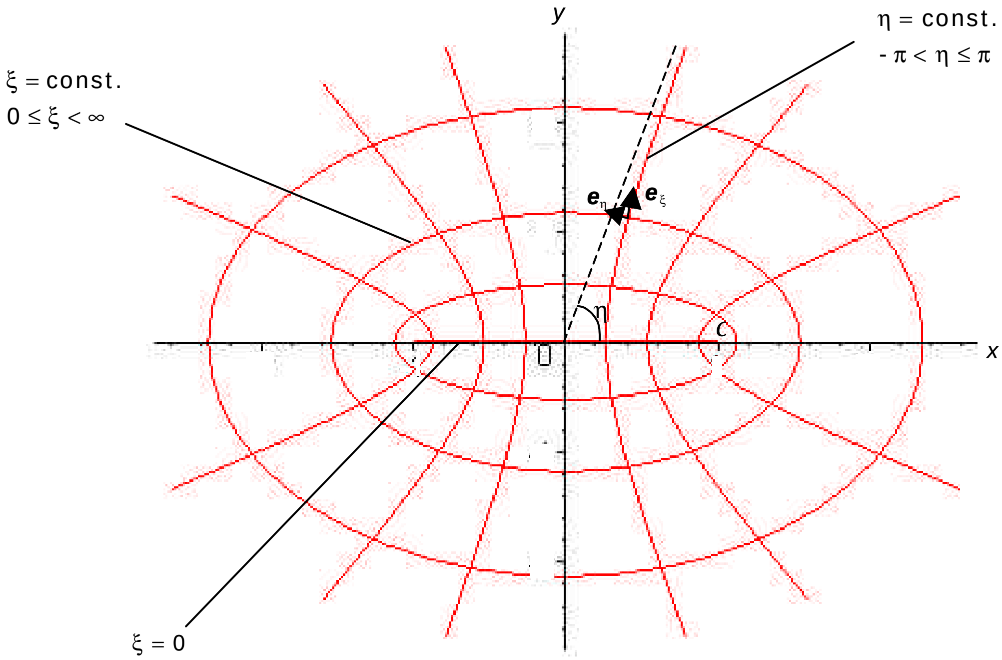

Fig 4 Elliptic coordinates. Aerofoil surface is at $\xi = \xi_0 = 0.5\ln((a+b)/(a-b))$. Foci are at $\pm c$ with $c = \sqrt{a^2 - b^2}$.

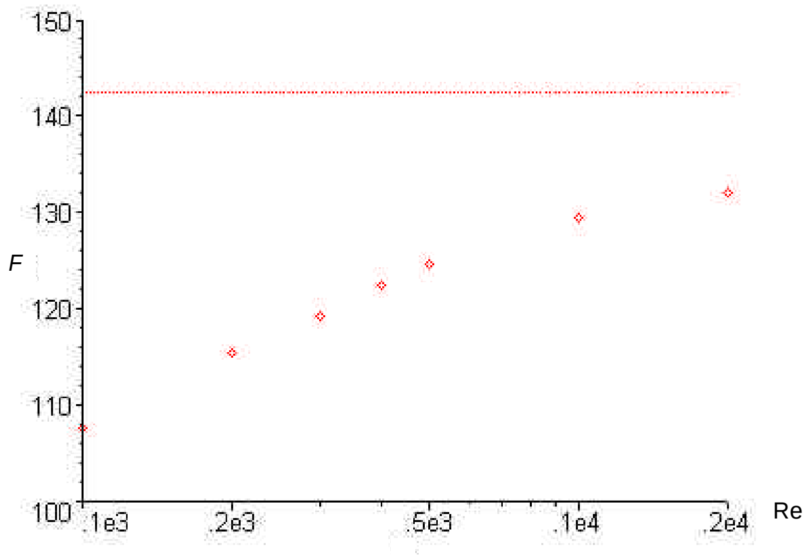

Fig 5 $F(\mathrm{Re}, a/b)$ versus Re for laminar flow with $a/b = 5$.
Points are CFD values. Dotted line is for potential flow.