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ON SOME DIFFICULTIES IN THE MAXIMUM LIKELIHOOD ESTIMATION

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Abstract: Two-parameters double exponential distribution is presented. Maximum likelihood method is applied to estimate the parameters of the distribution. It seems that the maximum likelihood estimates of the double exponential distribution exist only for rather specific collections of data.
Keywords: double exponential distribution, maximum likelihood estimation,

1 INTRODUCTION

Books on statistics discuss double exponential distribution not often. But still it may serve as a candidate distribution for modeling some random phenomenon resulting in a continuous random variable. Then the two parameters of the distribution have to be estimated from a collection of data. Using maximum likelihood method for parameter estimation is the right choice since the maximum likelihood estimator has nice properties. In the case considered we get two likelihood equations which data dependent solution (the one that gives the maximum value of the likelihood function) is the maximum likelihood estimate of the distribution parameters. Unfortunately, for the double exponential distribution getting the solution of the likelihood equations being the estimate of the distribution's parameters is often impossible.

In the paper the collection of data for which the aforementioned trouble occurred is presented. The cause of the trouble is recognized. It is pointed out that in the trouble condition the formulas of the maximum likelihood estimators of the double exponential distribution parameters suggest to consider the (ordinary) exponential distribution as a candidate probabilistic model what is in agreement with phenomenal reasoning.

2 DOUBLE EXPONENTIAL DISTRIBUTION

Lloyd and Lipov [1] quote the double exponential distribution having probability density function (pdf) defined by

$$f(x) = \begin{cases} 0 & \text{for } x < 0 \\ \alpha\beta e^{\beta x} e^{-\alpha\left(e^{\beta x}-1\right)} & \text{for } x \geq 0 \end{cases} \tag{1}$$

where $\alpha>0$ and $\beta>0$ are the distribution's parameters.

Fig. 1 depicts double exponential pdf with different values of the distribution's parameters. For all plots in Fig. 1.a the parameter $\beta=1$, and in Fig. 1.b the parameter $\alpha=1$. For the double exponential distribution the parameter β is the parameter of scale.

Let us assume that we want to establish whether or not sample data have been generated by a process characterized by the double exponential distribution. We placed N observables into k class intervals. Then we have to estimate the value of the distribution's parameters. To achieve this end the maximum likelihood method was used.

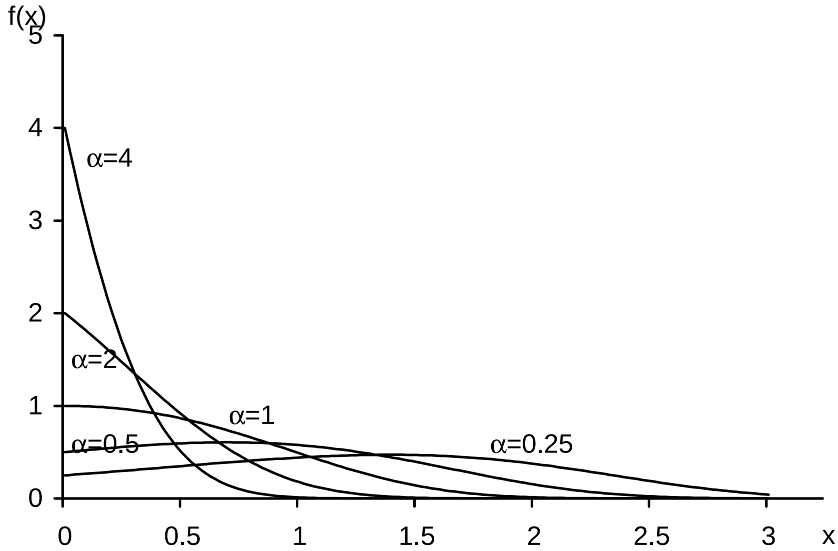
The likelihood function for the double exponential distribution takes the form

$$L(x_1, x_2, ..., x_k, \alpha, \beta) = \alpha^N \beta^N e^{\beta \sum_{i=1}^{k} n_i x_i} e^{-\alpha\left(\sum_{i=1}^{k} n_i e^{\beta x_i} - N\right)} \tag{2}$$

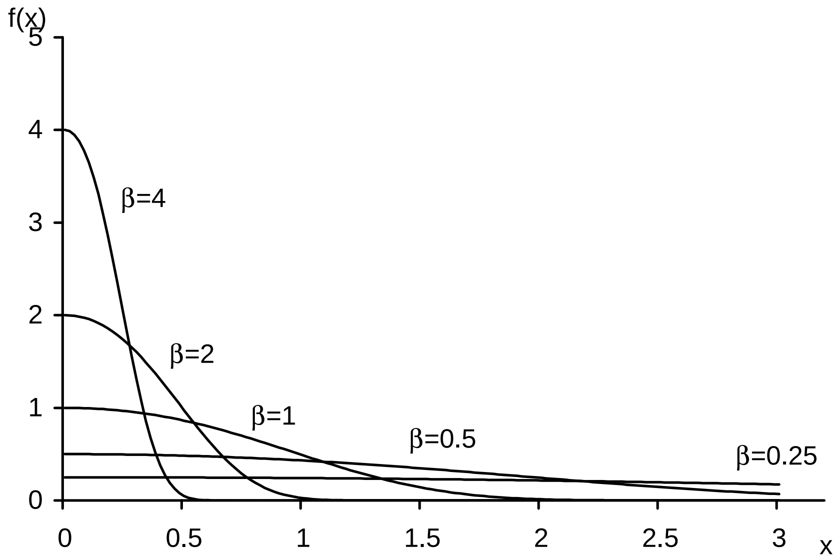
where i=1,2,...,k, n_i is the number of observables falling into the i-th class interval and x_i is the i-th class interval midpoint.

From (2) we have

$$\ln L = N \ln\alpha + N \ln\beta + \beta \sum_{i=1}^{k} n_i x_i - \alpha\left(\sum_{i=1}^{k} n_i e^{\beta x_i} - N\right) \tag{3}$$



a.



b.

Figure 1. Double exponential distribution: a. β=1; b. α=1

Taking derivative ln*L* with respects to α and β we get

$$\frac{\partial \ln L}{\partial \alpha} = \frac{N}{\alpha} - \sum_{i=1}^{k} n_i e^{\beta x_i} + N \qquad (4)$$

$$\frac{\partial \ln L}{\partial \beta} = \frac{N}{\beta} + \sum_{i=1}^{k} n_i x_i - \alpha \sum_{i=1}^{k} n_i x_i e^{\beta x_i} \qquad (5)$$

Equating (4) and (5) with zero after some manipulation we receive likelihood equations in the form

$$\hat{\alpha} = \frac{N}{\sum_{i=1}^{k} n_i e^{\hat{\beta} x_i} - N} \qquad (6)$$

$$\hat{\beta} = \frac{N\left(\sum_{i=1}^{k} n_i e^{\hat{\beta} x_i} - N\right)}{N\sum_{i=1}^{k} n_i x_i e^{\hat{\beta} x_i} - \left(\sum_{i=1}^{k} n_i x_i\right)\left(\sum_{i=1}^{k} n_i e^{\hat{\beta} x_i} - N\right)} \tag{7}$$

where $\hat{\alpha}$ and $\hat{\beta}$ are the estimators of the parameters α and β of the double exponential distribution.

3 ESTIMATION TRIALS

To check the correctness of the relations (6) and (7) determining the estimators $\hat{\alpha}$ and $\hat{\beta}$ the sample was taken from a population that was double exponential distributed with the parameters α=2.0 and β=1.0. The frequency distribution of this sample is given in Table 1. To solve (7) for estimate $\hat{\beta}$ a search procedure was applied. For assumed value of $\hat{\beta}=\beta_L$ the right side of (7) denoted as β_R was computed. The search stopped when the relative error $|\beta_L-\beta_R|/\beta_L$ was close to zero (within a reasonable accuracy). The estimate $\hat{\beta}$=1.073 was found and then from (6) $\hat{\alpha}$=1.8222.

Table 1. Double exponential distributed sample

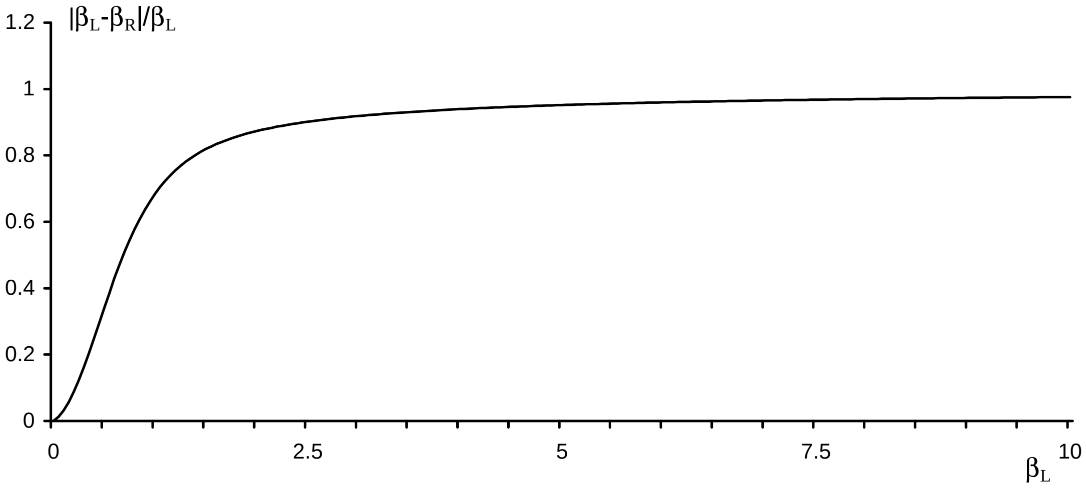
	x_i	n_i
1	0.05	569.00
2	0.15	504.00
3	0.25	436.00
4	0.35	368.00
5	0.45	301.00
6	0.55	239.00
7	0.65	184.00
8	0.75	136.00
9	0.85	96.00
10	0.95	65.00
11	1.05	41.00
12	1.15	25.00
13	1.25	14.00
14	1.35	7.00
15	1.45	3.00
16	1.55	1.00

Then for dozen of data collections received from an experiment on speech signal the double exponential distribution was considered as a candidate modeling amplitude distribution of the signal. Table 2 provides an example of the sample frequency distribution of quantized speech signal (in relative values).

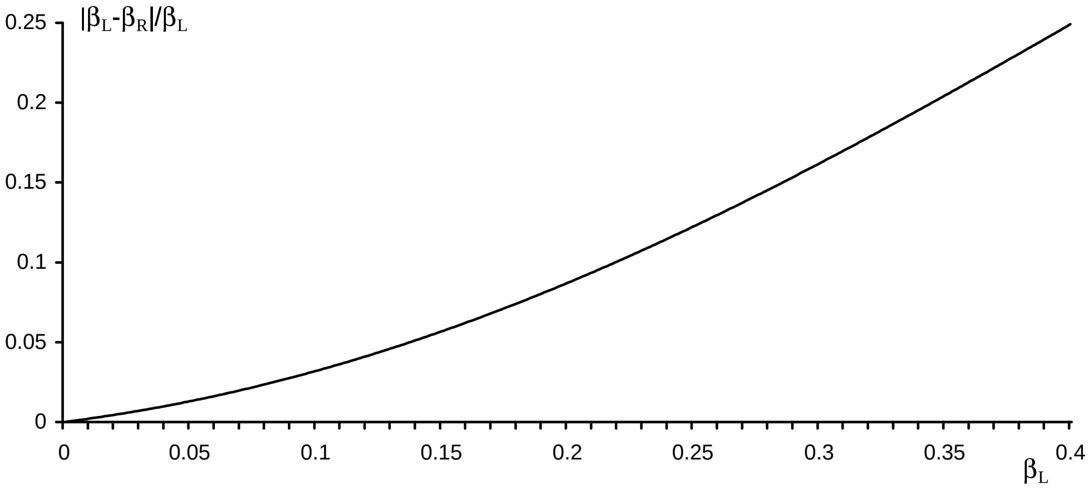
Table 2. Frequency distribution of quantized speech signal

	x_i	n_i
1	0.05	200
2	0.4	220
3	0.7	71
4	1.1	70
5	1.5	43
6	1.9	40
7	2.3	27
8	2.7	18
9	3.1	17
10	3.5	14
11	3.9	12
12	5.1	18

The search for $\hat{\beta}$ is shown in Figure 2. As one can see, the search failed. It happened for each data collection received from an experiment. It is worthwhile to notice that the same data collections produced maximum likelihood estimators of a Weibull distribution's parameters when the Weibull pdf was considered as a candidate model. (In fact a gamma distribution was chosen to model the phenomenon but it was the method of moments used to estimate the distribution's parameters).



a.



b.

Figure 2. The search for $\hat{\beta}$, data in Table 2 (b. part of Figure 2 is the zoom of part a. in the interval 0 β_L 0.4)

It seems that the estimation troubles originate in double exponential dependence of the pdf on data and the parameter β values. It makes the pdf of the double exponential distribution as well as its likelihood function extremely sensitive to data values when the distribution's parameters are estimated. It results in rather common event that the maximum of the likelihood function is located outside the domain of the distribution parameter's estimators, which of course have to be data dependent.

For different intermediate values of $\hat{\alpha}$ and $\hat{\beta}$ received when we tried to solve (6) and (7) a goodness-of-fit test was employed to determine if the data are consistent with the double exponential distribution. It was noticed that the value of chi-square statistic diminished with the decreasing $\hat{\beta}$ and growing $\hat{\alpha}$.

This fact suggests to take into consideration the limit of the product $\hat{\alpha}\,\hat{\beta}$ if $\hat{\beta} \to 0_+$. Using (6) and (7) we get

$$\hat{\gamma} = \lim_{\beta \to 0_+} \hat{\alpha}\hat{\beta} = \frac{N}{\sum_{i=1}^{k} n_i x_i} \tag{8}$$

which is the maximum likelihood estimator of the parameter of the (ordinary) exponential distribution. One can think that the maximum likelihood estimators of the double exponential distribution's parameters "suggest to use" exponential distribution as a candidate model in the case of estimation trouble, as being much less sensitive to the data values.

When the condition

$$\gamma = \lim_{\substack{\beta \to 0_+ \\ \alpha \to +\infty}} \alpha\beta \text{ and } \gamma > 0$$

is fulfilled then the double exponential distribution reduces to the exponential distribution [2].

4 CONCLUSION

It seems that maximum likelihood estimates of the double exponential distribution exist only for rather specific collections of data.

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