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PRIOR INFORMATION IN PRODUCT-RATIO MEASUREMENTS

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Abstract: In this paper we consider direct repeated measurements of a quantity R that is defined by the product $P_1^{n_1} \cdot \ldots \cdot P_m^{n_m}$ *(*$n_1, \ldots, n_m = \pm 1$*) of m other quantities. Bayes theorem is applied to obtain posterior estimates of R and* P_a *(a = 1,...,n) based on the measurements of R and prior information (estimates, associated uncertainties) about* P_a*. The paper extends a previous result [1] applicable to a calibration chain with only two elements to chains with arbitrary number of elements.*
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1 INTRODUCTION

By using Bayes theorem it is possible to incorporate into the final evaluation of a quantity not only the actual measured values but also information (usually in form of an estimate and associated uncertainty) about the quantity obtained from other sources prior to measurement. In a previous paper [1] an approach was developed to estimate a ratio of two quantities as well as the two quantities in the ratio. The estimation is based on direct measurements of the ratio and prior information about the two quantities. In the present paper this approach is applied to the more general case of a product of more than two quantities. The product may contain ratios. Repeated direct measurements of the product

$$R = P_1^{n_1} \cdot \ldots \cdot P_m^{n_m} \; (n_1, \ldots, n_m = \pm 1) \qquad (1)$$

yield measured values

$$R_1, \ldots, R_n. \qquad (2)$$

It is supposed that prior estimates

$$P_{oa} \; (1 \pm u_{rel} (P_{oa})) \qquad (3)$$

are available for the quantities

$$P_1, \ldots, P_m \, . \qquad (4)$$

The values (2) are supposed to be measured with the same relative uncertainties

$$u_{rel} (R_i) = u_{rel} \, . \qquad (5)$$

Also, the values (2) are supposed to be independent.

One can easily see that model (1) includes the case when the values (2) are obtained in measurements of ratio

$$R = N / D \, . \qquad (6)$$

This particular case, which has been considered in paper [1], corresponds, for example, to the following parameters

$$n_1 = +1, \, n_2 = -1, \, n_2 = \ldots = n_m = 0 \, . \qquad (7)$$

A linearization of the general task (1) will be performed (Section 2) to obtain the relative quantities and the Bayes theorem will be applied to the relative quantities (Section 3). As in the previous paper [1] the distributions in the following will be supposed to be Gaussian (see, also [2]).

2 LINEARIZATION OF THE TASK

The linearization of (1) around the prior values P_o suggests the introduction of the relative quantities

$$p_a = P_a / P_{0a} - 1, \; r = R / (P_{o1} \cdot \ldots \cdot P_{om}) - 1 \qquad (8)$$

as well as the new variables $q = n \; p$ with the result

$$r = q_1 + \ldots + q_m. \qquad (9)$$

instead of relation (1). The linearization leads also to relative measured values

$$r_1 = R_1 / (P_{o1} \cdot \ldots \cdot P_{om}) - 1, \ldots, r_n = R_n / (P_{o1} \cdot \ldots \cdot P_{om}) - 1 \qquad (10)$$

instead of the absolute measured values (2) as well as to relative prior estimates calculated according to (8)

$$q_{oa} \equiv n_a \, p_a \, (P_a = P_{oa}) \pm u_{rel} \, (P_{oa}) = 0 \pm u_{rel} (P_{oa}) \qquad (11)$$

instead of the absolute prior estimates (3).

Thus, we have prior information in the form of relative estimates and associated uncertainties (11) and measurement results which can be presented in the following analogous form

$$r_1 \pm u_{rel},\dots,r_n \pm u_{rel}\,. \tag{12}$$

To combine (11) and (12) the Bayes theorem will be applied.

3 APPLICATION OF BAYES THEOREM

The prior distribution function of the quantity q is given by expression

$$f_o(q_a) = \exp[-q_a^2\ /2\,s_{oa}^2\,]\Big/\sqrt{2ps_{oa}^2} \tag{13}$$

where $s_{oa} = u_{rel}(P_{oa})$.

We further assume the likelihood

$$f_L(r_1,\dots,r_n) = \exp\left[-\sum_i (r_i - r)^2\ /2s^2\right]\Big/\sqrt{2^n p^n s^{2n}} \tag{14}$$

where $s = u_{rel}(R_i)$.

According to Bayes theorem the posterior distribution function for q_1 , , q_n is given by the expression

$$f_{post}(q_1,\dots,q_m) = \frac{f_L(r_1,\dots,r_n)\, f_o(q_1)\cdot\ldots\cdot f_o(q_m)}{\int\ldots\int dq_1\ldots dq_m\, f_L(r_1,\dots,r_n)\, f_o(q_1)\cdot\ldots\cdot f_o(q_m)}. \tag{15}$$

From (13) and (14) the normal law follows for the posterior distribution of quantities $q_{1,\dots,}q_{m.}$ and for the prior distribution of sample $r_{1,\dots,}\ r_n$.

To find the explicit form of the distribution (15) we will follow to the scheme developed in [3]. This scheme is based on consideration of the two representations of the joint distribution of the quantities q_1 , , q_m and the sample $r_{1,\dots,}\ r_n$

$$f(q_1,\dots,q_m,r_1,\dots,r_n) = f_L(r_1,\dots,r_n\,|q_1,\dots,q_m)\,f_o(q_1)\cdot\ldots\cdot f_o(q_m) =$$

$$f_{post}(q_1,\dots,q_m\,|r_1,\dots,r_n)\,f_o(r_1,\dots,r_n)\,.$$

Straightforward calculation according to this scheme gives the posterior function in the form

$$f_{post}(q_1,\dots,q_m) \equiv f(\mathbf{q}) = \frac{1}{(2p)^{m/2}\,|\mathbf{V}|^{1/2}}\exp\left[-\frac{1}{2}(\mathbf{q}-\hat{\mathbf{q}})^T\,\mathbf{V}^{-1}(\mathbf{q}-\hat{\mathbf{q}})\right] \tag{16}$$

with the vectors

$$\mathbf{q} = (q_1,\dots,q_m)^T,\quad \hat{\mathbf{q}} = (\hat{q}_1,\dots,\hat{q}_m)^T. \tag{17}$$

Here the vector $\hat{\mathbf{q}}$ contains the expectations $\hat{q}_a$ which are given by the expression

$$\hat{q}_a = \frac{\bar{r}\Big/(s^2\ /n + \sum_{b\neq a} s_{ob}^2\,)}{1\Big/s_{oa}^2 + 1\Big/(s^2\ /n + \sum_{b\neq a} s_{ob}^2\,)} \tag{18}$$

where

$$\bar{r} = (r_1 + \ldots + r_n)/n\,. \tag{19}$$

The expression (18) can be interpreted as a weighted mean of the two values one of which is zero prior estimate (11) with the weight equal to $1/s_{oa}^2$ and the other is the combination

$$\bar{r} - \sum_{b\neq a} q_{ob} \equiv \bar{r} - 0 = \bar{r} \tag{20}$$

with the weight 1 / ($s^2\ /n + \sum_{b\neq a} s_{ob}^2$). It is clear from (18) that the value (20) plays the role of a measured value of q_a , which is to be weighted with the prior value (11) within the Bayesian approach.

The elements V_{ab} of the covariance matrix $\mathbf{V}$ for the posterior distribution (16) are

$$V_{ab} = d_{ab}\, s^{2}_{oa} - s^{2}_{oa} s^{2}_{ob} \;/[s^{2} / n + \sum_{g=1}^{m} s^{2}_{og}] . \tag{21}$$

From (21) the variance associated with the posterior estimate (18) is

$$V_{aa} = s^{2}_{oa} - s^{4}_{oa} \;/[s^{2} / n + \sum_{g=1}^{m} s^{2}_{og}] . \tag{22}$$

The expression (22) can be also transformed to the form

$$V_{aa} = \frac{1}{1/s^{2}_{oa} + 1/(s^{2} / n + \sum_{b \neq a} s^{2}_{ob})}$$

(23)

which corresponds to the presentation (18) of the posterior estimate as the weighted mean.

The prior distribution of the sample

$\mathbf{r} = (r_1, \ldots, r_n)^T$

has the same form as given in [1]

$$f_o(\mathbf{r}) = \frac{1}{(2p)^{n/2} |\mathbf{B}|^{1/2}} \exp\left(-\frac{1}{2} \mathbf{r}^T \mathbf{B}^{-1} \mathbf{r} \right), \tag{24}$$

however the covariance matrix $\mathbf{B}$ includes m variances

$$\mathbf{B} = s^{2}\mathbf{I} + (s^{2}_{o1} + \ldots + s^{2}_{om})\Pi . \tag{25}$$

Here $\mathbf{I}$ is $(m \times m)$-unit matrix and Π is \`unit systematic' matrix

$$\Pi = \begin{pmatrix} 1 & \ldots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \ldots & 1 \end{pmatrix}. \tag{26}$$

According to (9) the a posteriori estimate $\hat{r}$ of r can be obtained from (18) by simple summation.

The verification of the above expressions can be performed by straightforward calculation. For example, the exponent of the product $f_{post}(q_1, \ldots, q_m | r_1, \ldots, r_n)\, f_o(r_1, \ldots, r_n)$ when multiplied by "-2" gives

$$J_{right} = \left[(\mathbf{q} - \hat{\mathbf{q}})^T \mathbf{V}^{-1} (\mathbf{q} - \hat{\mathbf{q}}) \right] + \left[\mathbf{r}^T \mathbf{B}^{-1} \mathbf{r} \right] =$$

$$\left[\sum_{a,b}^{m} \left(q_a - \frac{n\bar{r} s^{2}_{oa}}{s^{2} + nq} \right) \left(d_{ab} / s^{2}_{oa} + n / s^{2} \right) \left(q_b - \frac{n\bar{r} s^{2}_{ob}}{s^{2} + nq} \right) \right] + \left[\frac{1}{s^{2}} \sum_{i,k}^{n} r_i \left(d_{ik} - \frac{q}{s^{2} + nq} \right) r_k \right] \tag{27}$$

where

$$q = \sum_{a}^{m} s^{2}_{oa} . \tag{28}$$

Then, further transformation gives

$$J_{right} = \sum_{a}^{m} \left(q_a - \frac{n\bar{r} s^{2}_{oa}}{s^{2} + nq} \right)^2 \frac{1}{s^{2}_{oa}} + \frac{n}{s^{2}} \left(\sum_{a}^{m} \left(q_a - \frac{n\bar{r} s^{2}_{oa}}{s^{2} + nq} \right) \right)^2 + \frac{1}{s^{2}} \sum_{i}^{n} r_i^{2} - \frac{n^{2} \bar{r}^{2} q}{s^{2}(s^{2} + nq)} .$$

and

$$J_{right} = \sum_{a} \frac{q_a^{2}}{s^{2}_{oa}} - \frac{2n\bar{r}r}{s^{2} + nq} + \frac{n^{2}\bar{r}^{2} q}{(s^{2} + nq)^{2}} + \frac{nr^{2}}{s^{2}} - \frac{2n^{2}\bar{r} r q}{s^{2}(s^{2} + nq)} +$$

$$\frac{n^3\bar{r}^2 q^2}{s^2(s^2+nq)^2}+\frac{1}{s^2}\sum_i^n r_i^2-\frac{n^2\bar{r}^2 q}{s^2(s^2+nq)} =$$

$$\frac{1}{s^2}\sum_{i=1}^n r_i^2-\frac{2n\bar{r}r}{s^2}+\frac{nr^2}{s^2}+\sum_{a=1}^m\frac{q_a^2}{s_{oa}^2}+$$

$$\bar{r}r\left[\frac{2n}{s^2}-\frac{2n}{s^2+nq}-\frac{2n^2 q}{s^2(s^2+nq)}\right]+$$

$$\bar{r}^2\left[\frac{n^2 q}{(s^2+nq)^2}+\frac{n^3 q^2}{s^2(s^2+nq)^2}-\frac{n^2 q}{s^2(s^2+nq)}\right].$$

Taking into account that the exponent of the product $f_L(r_1,\ldots,r_n|q_1,\ldots,q_m)\, f_o(q_1)\cdot\ldots\cdot f_o(q_m)$ multiplied by "-2". is

$$J_{left} = \frac{1}{s^2}\sum_{i=1}^n r_i^2-\frac{2n\bar{r}r}{s^2}+\frac{nr^2}{s^2}+\sum_{a=1}^m\frac{q_a^2}{s_{oa}^2} \qquad (29)$$

and the coefficients in the square brackets for $\bar{r}r$ and $\bar{r}^2$ are equal to zero, we have

$$J_{left} = J_{right}\ .$$

Thus, the equality of the exponents is valid. The equality of the prefactors can be similarly verified.

4 CONCLUSION

Expressions (18) constitute a generalization of the corresponding solution in [1] where there were only two terms. Similarly, the covariances given by (21) were calculated in [3] for a ratio of two quantities. The two expressions for the covariances have opposite sign since in the present paper the resulting relative quantity (9) is presented in form of a sum while the paper [3] dealt with a difference of two relative quantities corresponding to the numerator and the denominator.

Application of the obtained expressions to the description of a calibration process is similar to that given in [3] exept that here an arbitrary number of elements (resistors, etc.) can be involved in the calibration chain.

REFERENCES

[1] V.Tuninsky and W.Wöger, Prior information in ratio measurements, in: J.Halttunen (ed.), *Proceedings of the XIV IMEKO Congress "New measurements - challenges and visions"* (Tampere, 1-6 June 1997), Volume V, p.164-167.

[2] K.Weise and W.Wöger, A Bayesian theory of measurement uncertainty, *Meas. Sci. Technol.*, 4, (1993) 1-11.

[3] V.Tuninsky and W.Wöger, Bayesian approach to recalibration, *Metrologia*, **34**, (1997) 459-465.

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