REDUCING INTERVAL ARITHMETIC IN DYNAMIC ERROR EVALUATION

## *J. Jakubiec and K. Konopka*

Institute of Measurements and Automatics for Electrical Engineering,
The Silesian Technical University, 10 Akademicka Str., 44-100 Gliwice, Poland

Abstract. Reducing interval arithmetic enables to describe properties of error sources in the matrix form. It is especially significant when algorithms process long measuring result sequences. Changes in time of measured quantity cause dynamic errors in the analog elements of measuring chain. When signal is nonsinusoidal the dynamic error can be presented as a set of harmonics. Those harmonics should be composed in order to get the resultant error. The method described in the paper enables to represent a set of harmonics as a set of intervals. The interdependence of those intervals is determined by the harmonics phase shifts. The method enables to calculate the amplitude of the harmonics sum in an approximate but simple way. The dynamic error amplitude calculated in the way described above is interpreted as a partial dynamic uncertainty of the measuring results. It can be composed with another kinds of partial uncertainties by using reducing arithmetic that enable to determine the final processing uncertainty.

Keywords: reducing interval arithmetic, amplitude of nonsinusoidal dynamic error, dynamic uncertainty

1   INTRODUCTION

Let us consider analog-to-digital measuring chain presented in Fig. 1. It contains three kinds of processing: analog marked as A/A, analog-to-digital A/D that consists of sampling and quantization and digital D/D performed by algorithms. It is assumed that measured quantity $x(t)$ varies in time what causes dynamic errors arising in analog part of the chain. These errors can be decreased using a correction algorithm [1]. The rests of not corrected dynamic errors influence the accuracy of results at the output of the measuring chain, therefore they should be taken into account while final uncertainty is estimated.
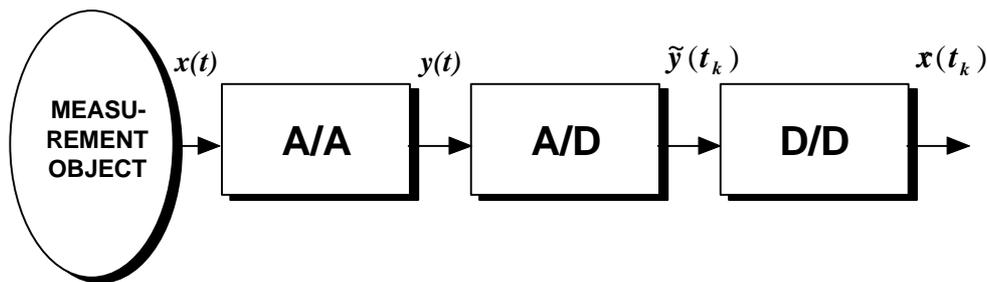
Figure 1. General structure of analog-to-digital measuring chain

It is convenient to interpret the measuring chain as a sampling converter [3]. Results $\hat{x}(t_k)$ at its output are estimates of instantaneous values of the input quantity $x(t)$ at the moments $t_k = kT_d$, where T_d is the sampling period, k is the number of the sampling moment, $k = 0,1,\dots$. The results are obtained on the basis of the quantized samples sequence $\tilde{y}(t_k)$ of the analog part output voltage $y(t)$. These samples are processed by a dynamic reconstruction algorithm which is built on the basis of the dynamic properties model of the analog part.

When measuring conditions are like described above, it is possible to determine amplitude of the dynamic error for any kind of the measured signal, obviously only when this signal is sampled frequently enough. It can be done by calculation of amplitudes and phases of the input signal

processing the sequence of samples $\hat{x}(t_k)$ by using Discrete Fourier Transform DFT. When dynamic properties of the analog part and the correction algorithm are known [1], the amplitude and phase of dynamic error for every harmonic can be determined. Summing up the harmonics enables to obtain amplitude of resultant dynamic error, however calculation of the amplitude is a complex task, therefore a simpler method is needed which is approximate but simple. The paper presents such method, which uses reducing interval arithmetic developed by the author of paper [2] in order to determine uncertainty of measuring data processing algorithms.

2 REDUCING INTERVAL ARITHMETIC

Reducing arithmetic is a kind of classical interval arithmetic [4] adapted to requirements of error sources properties modelling in analog-to-digital measuring chains with assumption that uncertainty is treated as a parameter of the error set [1]. It provides convenient mathematics apparatus to formalise description of uncertainty propagation in situation when analog, analog-to-digital, and digital processing is performed [2].

Interval arithmetic [4] is a mathematical apparatus enabling to perform operations on numerical intervals. The interval x is defined as a set of real numbers $\overline{x}$ such that $\underline{x} \leq \overline{x} \leq \overline{X}$, where $\underline{x}$ and $\overline{X}$ determine interval bounds and $\underline{x} \leq \overline{X}$. The bounds can be treated as interval parameters. One defines another parameters such as an interval midpoint denoted as $\hat{x} = \text{mid}(x) := \dfrac{x + \overline{x}}{2}$ and an interval radius $\text{rad}(x) := \dfrac{x - \overline{x}}{2}$. Taking those definitions into account the interval can be written as a range

$$x = [\hat{x} - \text{rad}(x), \hat{x} + \text{rad}(x)] = \hat{x} + [-\text{rad}(x), \text{rad}(x)] = \hat{x} \pm \text{rad}(x). \tag{1}$$

The final form of the expression above is a simplified interval notation as a sum of interval midpoint and so called unloaded interval which bounds are determined only by the radius.

It is assumed that the number of all unloaded intervals is limited and in general it is equal to *N*. These intervals form a set denoted **IR** in *N* dimensional space of real numbers and they are called primitive intervals. According to the relation (1) every interval *x* can be created by adding any real number interpreted as the midpoint to the unloaded interval.

In classical interval arithmetic operations are performed as combinations of the interval bounds [4]. Then, one chooses two terminal values from obtained in this way 4-element set. These values are treated as bounds of resultant interval. It means that when making operations on intervals one actually performs them on their bounds. Another interval parameters are determined on the basis of these bounds, so they are secondary parameters.

In the reducing arithmetic there is performed an inverse procedure – interval midpoint and radius are the prime parameters and the interval bounds are calculated on this basis. Taking into account the independence of prime parameters there can be executed separate operations on them and then results can form the interval according to the equation (1).

Let us assume that primitive unloaded interval radiuses form a set of *N* vectors $\bar{\text{rad}}_1(x), \ldots, \bar{\text{rad}}_N(x)$ which have common origin in *N* dimensional space of real numbers. Radiuses orientation in space is relative – it is determined by the vectors mutual orientation. Every pair of vectors $\bar{\text{rad}}_i(x)$ and $\bar{\text{rad}}_j(x)$, $i, j \leq N$ lies on a common plane. The angle g_{ij} between the vectors on this plane defines their mutual orientation in space.

The set of radiuses of the unloaded intervals can be presented as a column matrix of *N* elements called an interval radius vector

$$\mathbf{\bar{rad}}(\mathbf{x}) = [\bar{\text{rad}}_1(x) \quad \bar{\text{rad}}_2(x) \quad \cdots \quad \bar{\text{rad}}_N(x)]^T, \tag{2}$$

where T is a symbol of transposition.

Exterior product of this vector is defined as *NxN* dimensional matrix

$$\mathbf{\bar{rad}}(\mathbf{x}) \cdot \mathbf{\bar{rad}}(\mathbf{x})^T = \begin{bmatrix} \text{rad}_1(x) \cdot \text{rad}_1(x) \cdot r_{11} \ldots \text{rad}_1(x) \cdot \text{rad}_N(x) \cdot r_{1N} \\ \vdots \qquad\qquad \vdots \\ \text{rad}_N(x) \cdot \text{rad}_1(x) \cdot r_{N1} \ldots \text{rad}_N(x) \cdot \text{rad}_N(x) \cdot r_{NN} \end{bmatrix}. \tag{3}$$

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Every element of this matrix is a scalar product of corresponding pair of interval radiuses. An ith interval radius length is denoted as $\mathrm{rad}_i(x) = \left|\overline{\mathrm{rad}}_i(x)\right|, i = 1,\ldots,N$. Coefficient r_{ij} is called coherence factor of the ith and the jth radiuses, where

$$r_{ij} = \begin{cases} 1 & \text{for} \quad i = j \\ \cos g_{ij} & \text{for} \quad i \neq j \end{cases} \qquad i,j = 1,\ldots,N.$$

The matrix (3) determined for unitary interval radiuses is called coherence matrix and has a form:

$$\mathbf{R_{xx}} = \begin{bmatrix} 1 \, r_{12} \, r_{13} \ldots r_{1N} \\ r_{21} \, 1 \, r_{23} \, \ldots r_{2N} \\ \vdots \\ r_{N1} \, r_{N2} \, r_{N3} \ldots 1 \end{bmatrix}. \tag{4}$$

Let us define a multiple interval as a sum of real number and n unloaded intervals from **IR**, $n \leq N$

$$x = \hat{x} \pm \mathrm{rad}_1(x) \pm \mathrm{rad}_2(x) \pm \ldots \pm \mathrm{rad}_n(x). \tag{5}$$

It is assumed that multiple interval radius is a vector which is a sum of partial intervals radiuses. In order to calculate its length the following equation can be used

$$\mathrm{rad}(x) = \left|\mathbf{rad(x)}\right| = \sqrt{\mathbf{rad(x)}^T \cdot \mathbf{R_{xnxn}} \cdot \mathbf{rad(x)}}. \tag{6}$$

where $\mathbf{rad(x)}$ is n dimensional vector which elements are the partial radiuses lengths and $\mathbf{R_{xnxn}}$ is a submatrix of $\mathbf{R_{xx}}$. The set of two primitive intervals i and j is the simplest case that can be considered. In this case, the resultant radius length can be written as

$$\mathrm{rad}(x) = \sqrt{\begin{bmatrix} \mathrm{rad}_i(x) \\ \mathrm{rad}_j(x) \end{bmatrix}^T \begin{bmatrix} 1 \, r_{ij} \\ r_{ji} \, 1 \end{bmatrix} \begin{bmatrix} \mathrm{rad}_i(x) \\ \mathrm{rad}_j(x) \end{bmatrix}} = \sqrt{\mathrm{rad}_i^2(x) + \mathrm{rad}_j^2(x) + 2 \cdot \mathrm{rad}_i(x) \cdot \mathrm{rad}_j(x) \cdot r_{ij}}, \tag{7}$$

where $r_{ij} = r_{ji}$. The equation above indicates that $\mathrm{rad}(x) \leq \mathrm{rad}_i(x) + \mathrm{rad}_j(x)$ for any pair of indexes $i,j=1,\ldots,N, \ i \neq j$. The resultant interval radius length is equal to the sum of primitive intervals radiuses lengths when $r_{ij} = 1$.

3 TRANSFORMATION OF HARMONICS INTO SET OF UNLOADED INTERVALS

Let us assume that measured signal can be presented as a sum of N harmonics

$$x(t) = X_1 \sin(wt + a_1) + X_2 \sin(2wt + a_2) + \ldots + X_N \sin(Nwt + a_N), \tag{8}$$

where $X_1, X_2, \ldots, X_N$ are the amplitudes of successive harmonics and $a_1, a_2, \ldots, a_N$ are their phase shifts in relation to common original point on the time axis. Processing this signal by an analog transducer causes arising dynamic error which can be written as

$$e(t) = E_1 \sin(wt + j_1) + E_2 \sin(2wt + j_2) + \ldots + E_N \sin(Nwt + j_N). \tag{9}$$

Input signal parameters and the transmittance of the transducer define the components of the error. When the input signal is sinusoidal, instantaneous values of the dynamic error change in symmetrical range and the error amplitude determines its bounds. Therefore, it is assumed that the uncertainty caused by the dynamic error is equal to its amplitude [1]. When the input signal is not sinusoidal the problem is more sophisticated. In many harmonics compositions the resultant signal maximum and minimum have different absolute values. In further considerations it is assumed that the uncertainty of the nonsinusoidal signal is equal to the maximum value of error (9) modulus:

$$\Delta_d = \left|e(t)\right|_{\max} = \left|e(t)\right| \text{ for } t = t_{\max}, \tag{10}$$

where $t_{\max}$ is a time value when the dynamic error (9) modulus is maximal.

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Determining such defined maximum of the signal of known harmonics is a difficult task. Therefore the approximate procedure is searched to simplify calculations. In the situation presented the task can be formulated as follows. The harmonics amplitudes determine partial uncertainties connected with them. Such representation of harmonics in **IR** space is searched that the length of the resultant vector is equal to the maximum value of the harmonics composition (10). It is assumed that angles between the interval radiuses are proportional to the difference of the phase shifts of corresponding harmonics. Therefore, the following transformation is sought

$$\mathbf{G} = \mathbf{A} \cdot \mathbf{F} + \mathbf{B},\tag{11}$$

where **G**, **A, F** and **B** are *NxN* matrixes and

$$\mathbf{G} = \begin{bmatrix} g_{11} g_{12} \cdots g_{1N} \\ g_{21} g_{22} \cdots g_{2N} \\ \vdots \\ g_{N1} g_{N2} \cdots g_{NN} \end{bmatrix}, \qquad \mathbf{F} = \begin{bmatrix} j_{11} j_{12} \cdots j_{1N} \\ j_{21} j_{22} \cdots j_{2N} \\ \vdots \\ j_{N1} j_{N2} \cdots j_{NN} \end{bmatrix},$$

is the matrix of interval radius angle and the matrix of harmonics phase shift difference, and

$$\mathbf{A} = \begin{bmatrix} a_{11} \ a_{12} \dots a_{1N} \\ a_{21} \ a_{22} \dots a_{2N} \\ \vdots \\ a_{N1} \ a_{N2} \dots a_{NN} \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} b_{11} \ b_{12} \dots b_{1N} \\ b_{21} \ b_{22} \dots b_{2N} \\ \vdots \\ b_{N1} \ b_{N2} \dots b_{NN} \end{bmatrix},$$

are the matrixes of constant coefficients. Moreover

$$\left. \begin{array}{ll} j_{ij} = -j_{ji} = j_i - j_j; & g_{ij} = -g_{ji} \quad \text{for} \quad i \neq j \\ j_{ij} = 0 \quad ; & g_{ij} = 0 \quad \text{for} \quad i = j \end{array} \right\} i, j \leq N.\tag{12}$$

Directly from relation (12) one derives:

$$a_{ij} = 0 \quad \text{for} \quad i \neq j \quad \text{and} \quad b_{ij} = 0 \quad \text{for} \quad i = j.\tag{13}$$

When all phase shifts j_{ij} are equal to 0 the matrix $\mathbf{F} = \mathbf{0}$, so basing on (11), (12) and (13) one has

$$b_{ij} = -b_{ji} \quad \text{for} \quad i \neq j, \quad i, j \leq N.\tag{14}$$

From (11), (12) and (13) relations the interval radius angles can be determined for any phase shifts, for example for 1st and 2nd harmonic it is:

$$g_{12} = a_{11} j_{12} + b_{12} \quad ; \quad g_{21} = a_{22} j_{21} + b_{21}.\tag{15}$$

Basing on (12) and (14) one obtains $a_{11} = a_{22}$. Analogous reasoning for another g_{ij} angles one obtains the following relation:

$$a_{11} = a_{22} = \dots = a_{ii} = a\tag{16}$$

To determine the a coefficient one solves the equation (11) for such phase shift angles that all angles $g_{ij} = 0$. For example when taking into account only 3 harmonics one obtains the following expressions

$$a = -\frac{4}{p} b_{12} = -\frac{3}{p} b_{13} = -\frac{12}{p} b_{23},\tag{17}$$

from where it results that

$$b_{13} = \tfrac{4}{3} b_{12} ; b_{23} = \tfrac{1}{3} b_{12}.\tag{18}$$

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To determine b_{12} coefficient the situation is considered that only 2 harmonics are present and $j_{12} = 0$. In such a case $g_{12} = b_{12}$ and while the amplitude of this signal is determined for example in simulative way, one can calculate the coefficient as $b_{12} \cong 0.989504$.

Taking all above relations into account, the equation (11) for exemplary 3 harmonics has the form:

$$\begin{bmatrix} 0 & g_{12} & g_{13} \\ g_{21} & 0 & g_{23} \\ g_{31} & g_{32} & 0 \end{bmatrix} = \begin{bmatrix} -\dfrac{4}{p}b_{12} & 0 & 0 \\ 0 & -\dfrac{4}{p}b_{12} & 0 \\ 0 & 0 & -\dfrac{4}{p}b_{12} \end{bmatrix} \begin{bmatrix} 0 & j_{12} & j_{13} \\ j_{21} & 0 & j_{23} \\ j_{31} & j_{32} & 0 \end{bmatrix} + \begin{bmatrix} 0 & b_{12} & \dfrac{4}{3}b_{12} \\ -b_{12} & 0 & \dfrac{1}{3}b_{12} \\ -\dfrac{4}{3}b_{12} & -\dfrac{1}{3}b_{12} & 0 \end{bmatrix} \quad (19)$$

The dynamic uncertainty is then calculated as in the expression below

$$\Delta_d = \sqrt{ \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix}^T \begin{bmatrix} 1 & \cos g_{12} & \cos g_{13} \\ \cos g_{21} & 1 & \cos g_{23} \\ \cos g_{31} & \cos g_{32} & 1 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} } . \quad (20)$$

When all coefficients are found, one can compare the amplitude estimated using those coefficients with the one determined accurately, for instance in simulative way. Figure 2 presents an exemplary histogram of the difference between those amplitudes when phase shifts are random. This figure contains also graph presenting how the range containing those differences changes with the change of one harmonic amplitude.
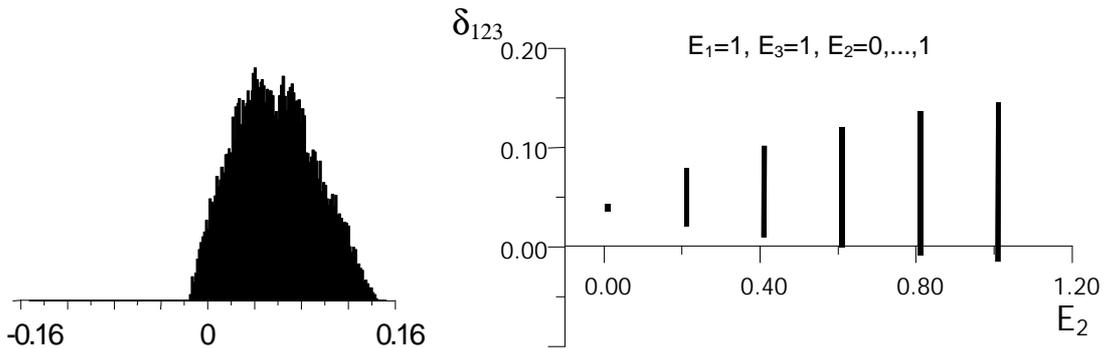
Figure 2. a) Exemplary histogram of the difference between the amplitudes calculated with the use of equation (20) and the accurate value, calculations for 3 harmonics and all amplitudes equal to 1, b) difference range changing with the 2nd harmonic amplitude

4   EXAMPLE

Let us consider measuring chain as on figure 3. It contains amplifier A, sample-and-hold circuit S/H and analog-to-digital converter A/D.
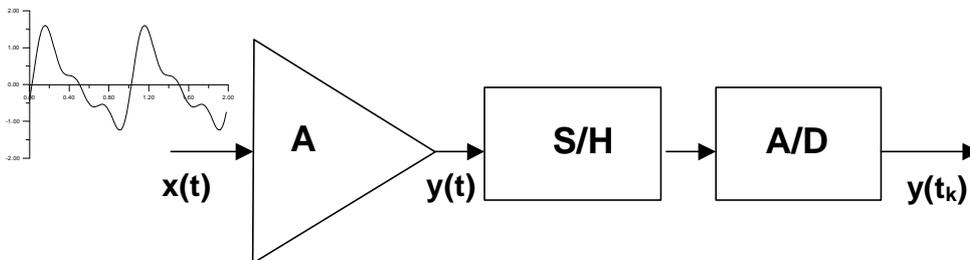
Figure 3. Exemplary measuring chain of instantaneous values of changing in time signal x(t)

Dynamic properties of the amplifier describes its transmittance

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$$S(j\boldsymbol{w}) = \frac{k_{u0}}{1 + j\dfrac{\boldsymbol{w}}{\boldsymbol{w}_0}} = \frac{Y(j\boldsymbol{w})}{X(j\boldsymbol{w})} \tag{21}$$

where $k_{u0} = 1$ is an amplification factor, $\boldsymbol{w}_0 = 2\boldsymbol{p} \cdot 10000\,Hz$ is a limiting frequency , $\boldsymbol{w} = 2\boldsymbol{p} \cdot 10\,Hz$ is an input signal frequency.

It is assumed that the input signal is:

$$x(t) = X_1 \sin(\boldsymbol{w}t + \boldsymbol{j}_1) + X_2 \sin(2\boldsymbol{w}t + \boldsymbol{j}_2) + X_3 \sin(3\boldsymbol{w}t + \boldsymbol{j}_3) \tag{22}$$

where $X_1 = 10V;\ X_2 = 5V;\ X_3 = 3,33V$ are the amplitudes of successive signal harmonics, and their phase shifts have values of $\boldsymbol{j}_1 = 0;\ \boldsymbol{j}_2 = -\frac{p}{7};\ \boldsymbol{j}_3 = -\frac{p}{4}$.

The dynamical error on the output of amplifier for sinusoidal signal is calculated as

$$e(j\boldsymbol{w}) = Y(j\boldsymbol{w}) - Y_{ideal}(j\boldsymbol{w}) = X(j\boldsymbol{w})[S(j\boldsymbol{w}) - k_{u0}] \tag{23}$$

where $Y_{ideal}(j\boldsymbol{w}) = X(j\boldsymbol{w}) \cdot k_{u0}$. In the considered situation

$$e(t) = E_1 \sin(\boldsymbol{w}t + \boldsymbol{f}_1) + E_2 \sin(2\boldsymbol{w}t + \boldsymbol{f}_2) + E_3 \sin(3\boldsymbol{w}t + \boldsymbol{f}_3) \tag{24}$$

where $E_1 \cong E_2 \cong E_3 = 0{,}01V$, $\boldsymbol{f}_1 = 1.5698, \boldsymbol{f}_2 = 1.5688, \boldsymbol{f}_3 = 1.5678$

The dynamic uncertainty calculated using the equation (20) is $\Delta_d \cong \pm 0.03\text{V}$.

It is assumed that A/D converter is an 8 bit one so quantization random errors have rectangular distribution in a range of $\pm 0.001953 \cdot 10V$. Therefore random uncertainty is $\Delta_r \cong \pm 0.02\text{V}$. In the situation presented, all coefficients in coherence matrix are equal to 1 so dynamic and random uncertainties are composed as in the following equation:

$$\Delta_w = \sqrt{\begin{bmatrix} \Delta_d \\ \Delta_r \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \Delta_d \\ \Delta_r \end{bmatrix}} = \pm 0.05\text{V} . \tag{25}$$

This final uncertainty was also determined in simulative way for comparison and its value is $\Delta_S = \pm 0.049V$.

5   FINAL REMARKS

The method presented enables to approximate amplitude of periodical signal which is composed of harmonics of known amplitudes and phase shifts. Considerations carried out above enable to state that maximum correct values differ from the ones calculated basing on expression (20) no more than several percent and introducing additional correction coefficients can reduce them. However, when the described method is used to determine uncertainty, its accuracy is sufficient. An advantage of the method is its simplicity and first of all its usability to determine final uncertainty which represents different errors sources in the measuring chain.

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AUTHORS: Prof. Jerzy Jakubiec, mgr eng. Krzysztof Konopka Institute of Measurements and Automatics for Electrical Engineering,The Silesian Technical University, 10 Akademicka Str., 44-100 Gliwice, Poland, Phone Int. ++48 32 37 25 05 Fax Int. ++48 32 37 20 34
e-mail: jakubiec@polsl.gliwice.pl , e-mail: kokon@wega.elektr.polsl.gliwice.pl