XVI IMEKO World Congress
Measurement - Supports Science - Improves Technology - Protects Environment ... and Provides Employment - Now and in the Future
Vienna, AUSTRIA, 2000, September 25-28

MODELING AND SIMULATION OF THE NON LINEAR EFFECTS IN ANALOG TO DIGITAL CONVERTERS

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Abstract: In this paper a novel approach for the direct computation of the probability density function (pdf) at the output of generic memory-less nonlinear processing blocks is derived. This method has been applied to the modeling of the THA-ADC chain. By using this approach, we are able to determine the exact behavior of the Signal to Noise Quantization Ratio (SQNR) of the THA-ADC chain for cases where the well-known Pseudo Quantization Noise (PQN) model do not give accurate results. The model can be easily improved to take into account for the ADC non idealities such as saturation and nonlinearity effects.
Keywords: probability density function, nonlinear processing blocks, ADC

1 INTRODUCTION

The new portable multimedia systems are characterized by very low-power absorption, very high throughput, high complexity. This fact imposes the use of sophisticated and accurate ADCs and DACs.

Moreover the market imposes shorter development times. Consequently the modeling of the ADC and DAC behaviour becomes more and more important. Starting from [1], in this paper we show a generalized procedure for the computation of the quantization error probability density functions (pdf) taking into account for the ADC non linearity. The analysis of the quantization effects is usually performed by using two different techniques. The first one is based on a statistical simulation. In this case the quantized processing system is stimulated with a significant input set. This input set models the statistic of the signals that are applied to the system during the actual operations. As a consequence, a great effort is devoted to the modeling due to the long simulations needed to obtain reliable estimations. A second possible approach considers the use of a linearized model of the quantizer plus a uniform error noise source of amplitude $1/q$ on the interval $[-q/2, q/2]$ (PQN model) [3], [4]. This model gives incorrect results if the conditions stated in [3], [4] on the characteristic function of the input pdf are not satisfied. The objective of this paper is the evaluation of the actual quantization noise pdf starting from a precomputed infinite arithmetic statistic. Due to the nonlinearity of the quantization process, the problem solution requires the analysis of the distortions introduced in the probability distribution by a piecewise linear system. We also provide a general procedure for this analysis. This procedure gives more accurate results with respect to those obtained by considering the white noise model. Moreover, the method is able to model some converter non idealities such as:

- Saturation effects
- Non linearity of the converter characteristic

The results of the proposed procedure match those obtained from a statistical (Montecarlo) simulation, while requiring a greatly reduced processing time. Such kind of analysis, for example, can show the exact behaviour of the SQNR (Signal to Noise Quantization Ratio) in the cases where the white noise model cannot be applied.

2 The New Algorithm

The new methodology for the evaluation of the quantization error, is based on some properties of the probability distribution functions (pdf). Given a random variable x the probability that it assumes a value in the range $\bar{x}$, $\bar{x} + d\bar{x}$ is obtained from its pdf $p_x(x)$ by using the following expression

$$P(x \in (\bar{x}, \bar{x} + d\bar{x})) = p_x(x)dx \tag{1}$$

If the random variable x is sent to the input of a function $f(x)$ to obtain a new variable $y = f(x)$. and the stochastic process x is *strict-sense stationary (SSS)* [2], the pdf of the new random variable y (namely, $p_y(y)$) is related to the pdf of the variable x through the following relation

$$p_y(y) = \frac{p_x(f^{-1}(y))}{\left(\frac{df}{dx}\right)_{x=f^{-1}(y)}} \tag{2}$$

The hypothesis on $f(x)$ concerns its derivability and the possibility to invert the function itself. If it cannot be inverted in the definition range, we can define a set of intervals where the function is monotonic. Therefore we obtain N intervals (quoted as Γ_i) numbered from *1* to *N*. For a given value of $\bar{y}$ we obtain a subset $\Phi(\bar{y})$ of the interval index such that the value $\bar{y}$ belongs to the coset of the function in the interval itself, i.e.,

$$\Phi(\bar{y}) = \{i : \exists\, \bar{x} \in \Gamma_i : f(\bar{x}) = \bar{y}\}$$

Let us consider a generic function $y = f(x)$. We have to compute the corresponding pdf $p_y(y)$ in the point $y = \bar{y}$. The set of the interval indexes that have $\bar{y}$ in the coset is $\Phi(\bar{y})$. The contribution of the interval Γ_i, with $i \in \Phi(\bar{y})$, to the pdf $p_y(\bar{y})$ is given by

$$p_y^{(i)}(\bar{y}) = \frac{p_x(f^{-1}(y))}{\left(\frac{df}{dx}\right)_{x=f^{-1}(y)}} \tag{3}$$

The final value can be obtained adding all the above values as follows

$$p_y(\bar{y}) = \sum_{i \in \Phi(\bar{y})} p_y^{(i)}(\bar{y}) \tag{4}$$

Starting from the above analysis, the algorithm can be summarized in the following four steps:

1. Definition of the monotonicity intervals Γ_i
2. Definition of the inverse function $f^{-1}(x)$ for each interval Γ_i.
3. Computation of equation (3) for each value $\bar{y}$ in the definition interval of y.
4. Computation of equation (4) for each value $\bar{y}$ to obtain the final value of the output pdf $p_y(\bar{y})$

In the next section we use the above procedure for the computation of the output pdf of different blocks. In particular we compute the output pdf of a THA having as input a sinusoidal signal and the pdf of the quantization error in an ideal analog to digital converter.

3 Modeling A THA-ADC Chain

Generally, the analog to digital conversion (ADC) is performed in two steps. The first one acquires a precise value of the input. This acquisition is driven by an ADC clock. The acquired value is held until the next clock cycle. These two operations are usually quoted as *sampling* and *holding*, and are performed by a *Sample-Hold* device. The second step is the conversion from analog to digital. While the first step is performed without information loss (if the conditions imposed from the Sampling Theorem are satisfied), the analog to digital converter gives, at the output, a quantized value of the input voltage.

3.1 THA output pdf computation

Let us consider an input periodic waveform of period T and suppose that the sample-hold system operates with a clock having period and phase that are uncorrelated with the period and phase of the input waveform. If we represent the waveform in the range $[0, T]$, the probability of selecting a sampling point in this interval is uniformly distributed. This corresponds to an input random variable with uniform distribution $p_t(t) = 1/T$ in the interval $[0, T]$. Consequently, the distribution of the sample-hold output is obtainable from the method discussed above using the function

$x(t) = A \cdot \sin\left(\frac{2p}{T}t\right)$ as a non linear block processing the input pdf (Fig. 1). The final expression for the pdf of x at the sample-hold output is

$$p_x(x) = \frac{1}{Ap\sqrt{1-\left(\frac{x}{A}\right)^2}} \tag{5}$$

The shape of $p_x(x)$ corresponds to the well known *Harmonic Process* . Its vertical asymptotes are placed at the points $x = \pm A$.

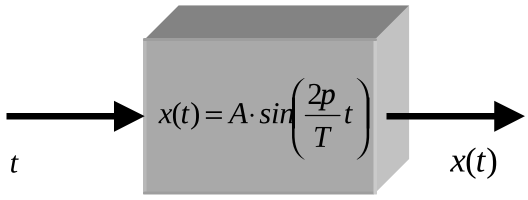

Figure 1 Nonlinear block modeling the THA in the case of sinusoidal input.

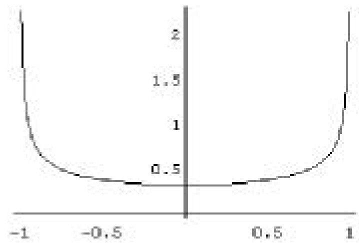

Figure 2 Probability density function for the sample-hold output (*A=1*).

3.2 Quantization noise pdf evaluation

In this section we discuss the noise effects of the analog to digital conversion process. The ADC effects are summarized in terms of the conversion error $e(x) = x - [x]_R$. Assuming a $Mbit$ bipolar rounding converter, taking into account that the index $k \in \left[-2^{Mbit-1}, 2^{Mbit-1}\right]$ corresponding to an analog interval from -1 to 1 where $q = 1/2^{Mbit}$, and introducing the triangular function $r_q(x)$ defined as $r_q(x) = x$ for $-q/2 \le x \le q/2$ and $r_q(x) = 0$ for $|x| > q/2$, we can derive, for $e(x)$, the following expression

$$e = f(x) = \sum_{k=-N}^{N} r_q(x - kq) \tag{6}$$

The function is represented in Fig. 3.

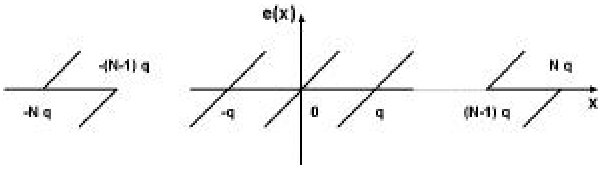

Figure 3. Error Function *e(x)* for linear quantization

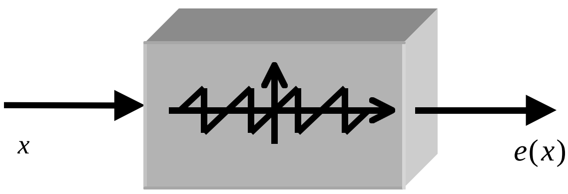

Figure 4. Non linear block modeling the quantization noise.

The pdf of e can be computed using the expressions (3) and (4) considering the transformation block shown in Fig. (4). This function has *2N + 1* intervals Γ_i where the inverse $x = f^{-1}(e)$ exists. Moreover, in each Γ_i we have $\frac{df}{dx} = 1$, therefore the pdf of the error conversion is obtained from

$$p_e(e) = \sum_{k=-N}^{N} p_e^{(i)}(e) \tag{7}$$

The terms $p_e^{(i)}$ in (7) are defined in Table 1.

Table 1.

For $\lvert i \rvert, N$	$p_e^{(i)}(e) =$	$p_x(e+iq)$	
For $i=-N$	$p_e^{(-N)}(e) =$	$p_x(e-Nq)$ 0	If $e>0$ If $e<0$
For $i=N$	$p_e^{(N)}(e) =$	0 $p_x(e+Nq)$	If $e>0$ If $e<0$

The above procedure procedure has been applied to a 6 bit bipolar ADC having an input signal obtained by sampling a sinusoidal signal whose pdf, $p_x(x)$, is shown in Fig. (2). The resulting pdf of the quantization error is shown in Fig. (5). In this case we assume $A=1$, i.e., a full input dynamic range. It is interesting to note that the shape of $p_e(e)$ depends on the relative positions of the vertical asymptotes in Fig. (2), i.e., on the input amplitude A and the bounds of the quantization interval q. For example, for a 6 bit ADC and A=0.19=12.1*q we obtain the pdf shown in Fig. (6). This figure shows the pdf's obtained both from the procedure above described and from the more computationally expensive statistical approach.

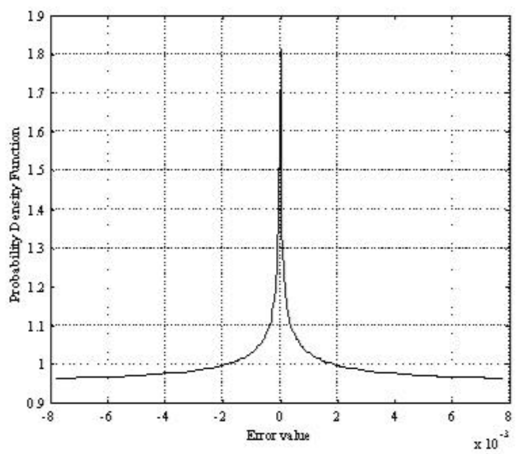

Figure 5. Probability density function of the quantization error $p_e(e)$ for *A=1 Mbit=6*

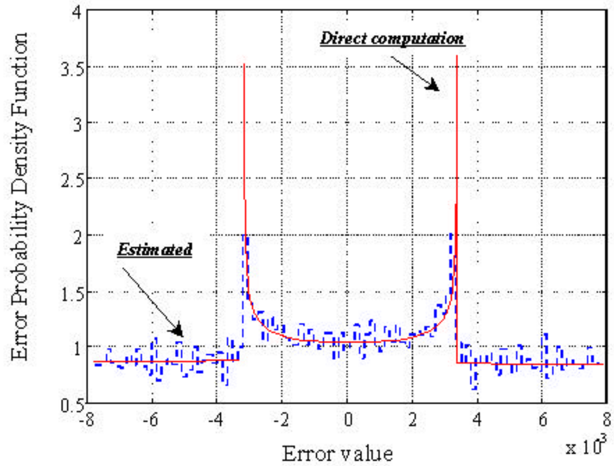

Figure 6. Probability density function of the quantization error $p_e(e)$ for *A=0.19=12.1*q Mbit=6* (direct computation and estimation by Montecarlo simulation).

As a second example, we evaluate the pdf of the quantization error for a nonlinear quantizer. We have modeled this quantizer as a cascade of a block that computes the function $\log_{10}$ of the input and a linear quantizer as described above. In this case the quantization intervals are characterized by three points

1. the left bound (corresponding to the analog value $10^{iq-q/2}$)
2. the central point, that gives the value of the output code (corresponding to the analog value 10^{iq})
3. the right bound (corresponding to the analog value $10^{iq+q/2}$)

The quantization error is equal to 0 in the central point of the interval and changes linearly with the input value. For this kind of converter, the intervals are not uniform and for a given error value ***e*** the set Φ does not contain all the intervals. Since the interval width grows with the index ***i***, (in fact $w_i = 10^{iq}(10^{q/2} - 10^{-q/2})$), large error values are given only by the intervals with large values of ***i***. This fact modifies the pdf of the error with respect of the pdf of the input variable. Let us consider for example the error quantization effects of a nonlinear converter whose input has a uniform distribution in the interval from 10^{-1+q} (the minimum value of the input) to 10^{1-q} (the maximum value). For the function $y = \log_{10}(x)$ and using a 3 bit ADC we obtain the pdf of the error shown in Fig. (7) (direct and simulative approach). This figure shows the effect of the different widths of the quantization intervals on the quantization error. In order to validate the obtained results, in the same figure we report the pdf estimated from a Montecarlo simulation. We can notice the good matching of over analytical results. The simulation required a large number of samples in order to obtain a sufficiently precise statistics. Consequently its computation required a larger CPU time in comparison with over proposed analytical method.

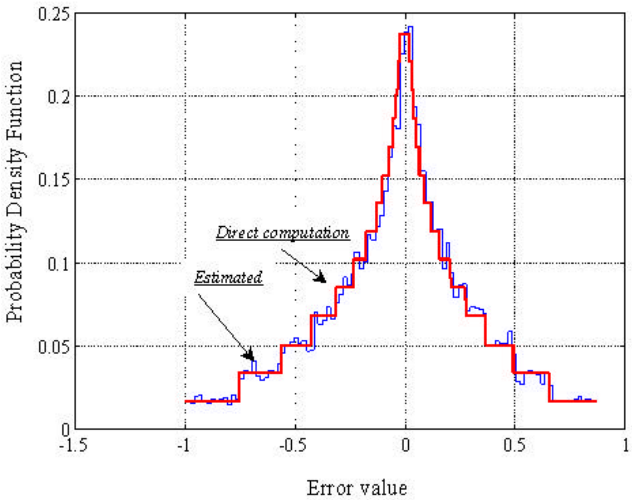

Figure 7. Error pdf for a quantizer with an input pdf obtained by a compressor *Nbit=3*, *A=7 q* (direct computation and estimation by Montecarlo simulation

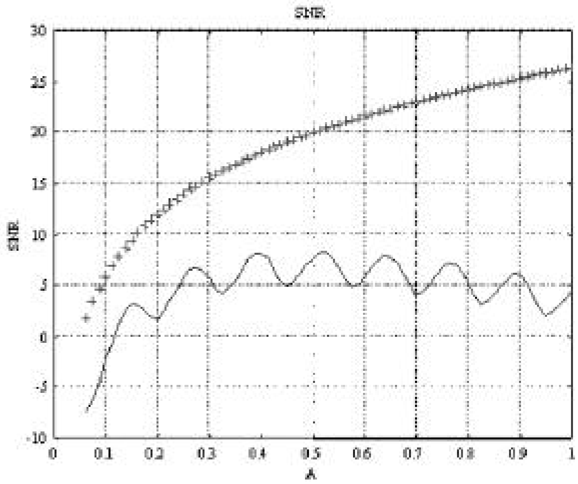

Figure 8. SQNR versus the input amplitude for a 8 bit ADC (PQN model ++, true SQNR ---)

3.3 Effects on the SQNR behaviour

It is well known that the quantization noise induced by the non linear behaviour of the ADC can be modeled by using the PQN (Pseudo Quantization Noise) model introduced by Widrow [3]. This model works well if the characteristic function of the input pdf satisfies the conditions stated in the QTI and or QTII (Quantization Theorem I and II) or the milder conditions stated in [4]. A simple rule of thumb is that the variance of the input process must be greater than the quantization step. This hypothesis is often not true in very high speed acquisition systems where the number of conversion bits is necessarily small (3-4). In this case, it is important to evaluate exactly of the power of the error signal to obtain the real behaviour of the SQNR. The above procedure permits a fast and accurate evaluation of the shape of $p(e)$ and consequently of the SQNR. Another case in which the PQN model does not guarantee conservative estimations of the SQNR is the logaritmic compression system described above. Fig. (8) show the real SQNR and that obtained from the PQN for a 3 bit ADC. The input pdf is obtained by using a compressor with a sinusoid as a input

3.4 The saturation effects

The ADC maps the input dynamic range to a finite number of output codes, consequently, if the input signal is over the dynamic range, the ADC will continue to assign to the input voltage the same output code. This fact implies that the conversion error grows in the extreme regions of the error characteristic as shown in Fig. 9. The method previously described for the fast computation of the pdf of the quantization error can be modified in order to take into account of the saturations regions in the static quantization error characteristic. For example the shape of the pdf of the quantization error for different values of the input voltage over the maximum value admitted by the ADC is shown in Fig. (10). In Fig. (11) a comparison of the of the SQNR behaviour of the PQN model with respect to that computed by using the proposed model is shown. This picture shows the fast decay of the SQNR when we are in thesaturation regions. The same approach can also be utilized to take into account of the ADC non linearities.

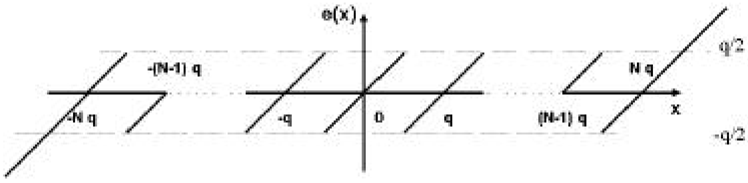

Figure 9. Error Characteristic for a ADC with saturation

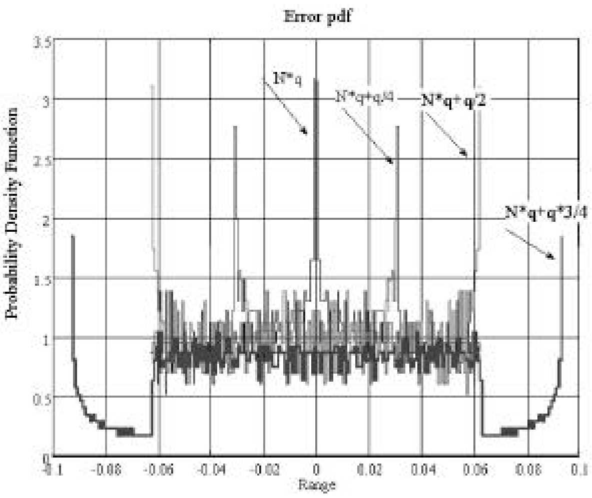

Figure 10. pdf of the quantization error for different amplitudes of the input voltage and in the saturation zone

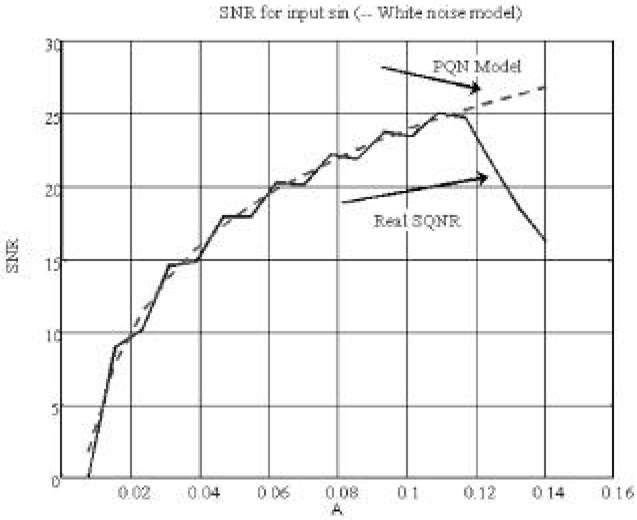

Figure 11. SQNR evaluated for the PQN model and our computation (M=4 bits)

4 Conclusions

In this paper a novel approach for the direct computation of the probability density function (pdf) at the output of generic memory-less nonlinear processing blocks has been shown. This method is as general as statistical simulation, is exact and requires much smaller computation time. The proposed methodology has been applied as an example to the THA-ADC chain. The exact behaviour of quantization error pdf has been studied both in the case of saturation effects and quantization step non linearities.

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This research was supported by CNR and Cadence Design Systems, Inc