XVI IMEKO World Congress
Measurement - Supports Science - Improves Technology - Protects Environment ... and Provides Employment - Now and in the Future
Vienna, AUSTRIA, 2000, September 25-28

LEAST-SQUARES VERSUS MINIMUM-ZONE FORM DEVIATIONS

D. Janecki and S. Adamczak
Center for Laser Technology of Metals and Faculty of Mechanical Engineering
Kielce University of Technology, Poland

Abstract: This work deals with theoretical research into the occurrence of maximum differences between form deviations obtained for two types of reference profiles - least squares profile and minimum zone profile. The investigations concerned roundness, sphereness, (two- and three-dimensional) straightness, flatness and cylindricity profiles. Examples of interrupted and filtered profiles were also analysed.
Keywords: Form deviation, Least-squares deviation, Minimum-zone deviation

1 INTRODUCTION

Form errors of measured profiles can be evaluated by means of various types of reference profiles [1,2]. In order to determine a roundness profile, for example, one can apply the minimum circumscribed circle, the maximum inscribed circle, least-squares and minimum-zone circles. The least-square and the minimum-zone reference profiles are of particular importance, the latter assuring the least possible form deviation defined in a most intuitive way. However, the determination of the minimum-zone deviation usually demands implementation of complicated and time-consuming optimisation procedures. Being of an iterative character, these procedures are known to provide approximate or even incorrect results, as in the case of existence of many local minima. On the other hand, the application of the least-squares reference profiles enables analytical determination of formulas for the deviation value and, in consequence, simplification of calculation procedures. Hence computer systems that are used in the measurement of form profiles frequently employ the least-square reference profiles, despite the fact that deviations determined on the basis of them have higher values.

It is generally agreed that the least-squares deviation may be a dozen or several dozen per cent higher than the minimum-zone deviation. The paper is concerned with theoretical determination of the maximum difference between the least-squares deviation and the minimum-zone deviation for various form types. The results are provided for roundness, sphereness, (two- and three-dimensional) straightness, flatness and cylindricity profiles. Interrupted and filtered profiles are also considered.

2 DEFINITIONS AND PROBLEM STATEMENT

Let us assume that the real profile of the measured object P is in the form of a function of the position variable θ defined on a certain subset Θ of the position space, that is:

$$P = P(q) : p \in \Theta . \quad (1)$$

Both the type of variable q and the form of set Θ depend on the form of the measured object. For instance, when measuring roundness of a cylinder-shaped object in a certain cross-section, it is convenient to assume that the variable q determines the angular position (later referred to as j), and Θ is the interval $[0,2p)$ of the space of real numbers. However, if we are interested in the whole area of the cylinder, we can assume that $q = (j, h)$, where φ is the angular position, and the variable h determines the distance between the cylinder section and its symmetry centre. In this case $\Theta = [0,2p) \times [-H, H]$, where H is half of the cylinder height.

Let $P_n(q, p)$ be a certain class of nominal profiles, where p represents parameters determining the size and position of the nominal profile in relation to the real profile. If P is a roundness profile, the class of nominal profiles is a set of circles. The parameter p represents the co-ordinates of the circle centre and its radius. A form deviation in relation to a selected nominal profile is defined as

$$\Delta P = \max_{q \in \Theta}(P(q) - P_n(q, p)) - \min_{q \in \Theta}(P(q) - P_n(q, p)) . \quad (2)$$

The least-squares reference profile can be defined as the nominal profile $P_n(q, p_{LS})$, for which the integral of squared difference between the real and the nominal profiles

$$J = \frac{1}{2}\int_{\Theta}\left(P(q) - P_n(q,p)\right)^2 dq \tag{3}$$

reaches minimum. The parameters p_{LS} defining the least-squares reference profile can be easily determined from the necessary conditions of optimality by equating the partial derivatives $\partial J/\partial p$ zero. The real profile deviation in relation to the least-squares nominal profile will be denoted by ΔP_{LS}.

The minimum-zone reference profile is defined as the nominal profile adjacent to the real profile for which deviation (2) reaches minimum. This deviation will be denoted by ΔP_{MZ}.

From the definition it follows, that the profile deviations ΔP_{MZ} and ΔP_{LS} satisfy the relationship

$$\Delta P_{MZ} \le \Delta P_{LS}\,. \tag{4}$$

The aim of this paper is to determine in s theoretical manner the maximum possible ratio of the least-squares deviation and the minimum-zone deviation of a given profile. In other words, for a desired nominal form of the measured profile, the minimum number κ has to be determined for which the relationship

$$\Delta P_{LS} \le k \cdot \Delta P_{MZ} \tag{5}$$

is always true. Hence the number κ defines how many times, in the worst-case, the least-squares deviation can be greater than the minimum-zone deviation. The results are provided for the roundness, sphereness, (two- and three-dimensional) straightness and cylindricity profiles. Interrupted and filtered profiles are also considered.

The analysis is carried out under the additional, in practice reasonable, assumption that the deviation ΔP_{MZ} is much smaller than the real profile dimension. Strictly speaking, we will find the limit of the coefficient κ when ΔP_{MZ} converges to zero.

3 ROUNDNESS DEVIATION

Let $j \in [0,2p)$ be the angular position and let $P(j)$ stand for a roundness profile in the polar coordinate system. Assume that for every j

$$P_{\min} \le P(j) \le P_{\max} \tag{6}$$

for certain $P_{\min}$ and $P_{\max}$. Assuming that the roundness profile is much smaller than the nominal radius of the measured object, the equation of the reference circle can be represented in the polar system as

$$P_n(j, a, b, R) = P_o + a\cos j + b\sin j \tag{7}$$

where R is the circle radius and a and b are the circle centre co-ordinates. The roundness deviation determined in relation to the reference circle fulfils the condition

$$\Delta P \le P_{\max} - P_{\min} + 2r\,, \tag{8}$$

where $r = \sqrt{a^2 + b^2}$ is the distance between the reference circle centre and the origin of the co-ordinate system circle. Indeed, the maximum profile peak and the maximum profile valley determined in relation to the reference circle are

$$P = \max_j (P(j) - P_o - a\cos j - b\sin j)) \le P_{\max} - P_o + r\,, \tag{9}$$

$$V = \min_j (-P(j) + P_o + a\cos j + b\sin j)) \le P_{\min} + P_o - r\,. \tag{10}$$

Hence we obtain the desired property (8). Let us estimate an upper bound of r for the least square circle. It should be noted that because of the symmetry we can assume that $b = 0$. It is known that the distance r is equal to the coefficient of the first harmonic of the profile expansion into the Fourier series, that is

$$r = a = \frac{1}{p}\int_{-p}^{p} P(j)\cos j\, dj\,. \tag{11}$$

Applying the inequality (6), we obtain

$$\begin{aligned} r &= \frac{1}{p}\left(\int_{-p/2}^{p/2} P(j)\cos j\, dj + \int_{p/2}^{3p/2} P(j)\cos j\, dj\right) \\ &\le \frac{1}{p}\left(\int_{-p/2}^{p/2} P_{\max}\cos j\, dj + \int_{p/2}^{3p/2} P_{\min}\cos j\, dj\right) = \frac{2}{p}(P_{\max} - P_{\min}) \end{aligned} \tag{12}$$

that is

$$\Delta P_{LS} \leq \left(1+\frac{4}{p}\right)(P_{\max} - P_{\min})\,. \tag{13}$$

It should be noted that we get the equality in (13) if

$$P(j) = \begin{cases} P_{\max} \text{ when } j \in (-p/2, p/2) \\ P_{\min} \text{ otherwise} \end{cases}. \tag{14}$$

But, for the above defined profile, the minimum-zone deviation is actually smaller than $P_{\max} - P_{\min}$, so in inequality (13), $P_{\max} - P_{\min}$ cannot be replaced with ΔP_{MZ}. However, the profile $P(j)$ defined by (14) can be slightly modified so that the value of the least-squares deviation ΔP_{LS} will not change substantially and the minimum zone deviation will be exactly equal to

$$\Delta P_{MZ} = P_{\max} - P_{\min}\,. \tag{15}$$

Let us assume that

$$P(j) = \begin{cases} P_{\max} \text{ when } j \in (-e, p/2) \cup (-p-e, -p+e) \cup (-3p/2, 2p-e) \\ P_{\min} \text{ otherwise} \end{cases} \tag{16}$$

for a small number ε. It should be noted that the profiles (14) and (16) overlap, except for some small fragments lying in the range $(-e, e)$ and $(-p-e, -p+e)$. Therefore, the value of the least-squares deviation will not change much if the parameter e is small enough. On the other hand, for the profile (16) we have (15). Thus, the following theorem has been proved:

Theorem 1: *If ΔP_{MZ} and ΔP_{LS} stand for the minimum-zone deviation and the least-squares deviation respectively, then*

$$\Delta P_{LS} \leq \left(1+\frac{4}{p}\right)\Delta P_{MZ}\,. \tag{17}$$

The obtained evaluation is the best possible in the sense that the number $1+4/p$ on the right side of inequality (17) cannot be replaced by a smaller one.

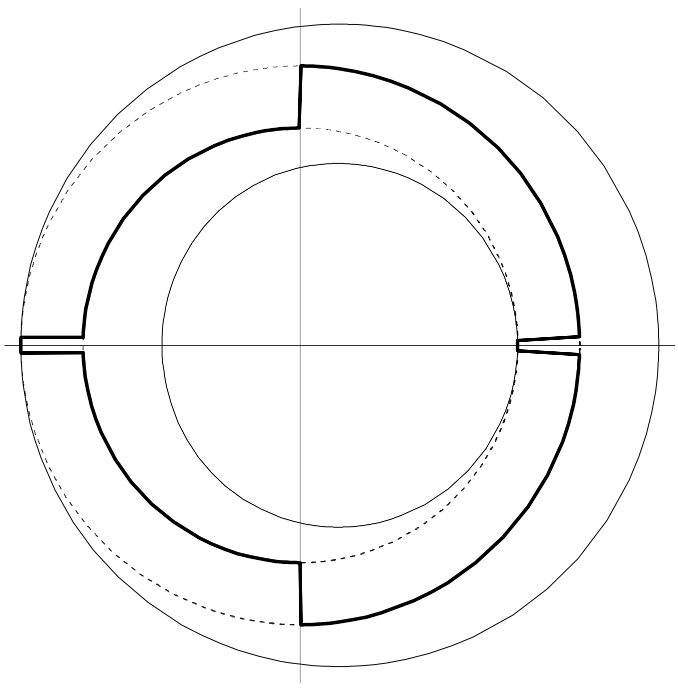

Figure 1. Roundness profile defined by (16) with $e = 0.03$ (solid heavy line), the circles inscribed and circumscribed about the profile with the centre in the centre of the mean circle (solid line) and the circles inscribed and circumscribed about the profile with the centre in the centre of the minimum-zone circle (broken line).

Note, that $1+4/p \approx 2.27$. Therefore, for some profiles, the least-squares deviation may be over twice as big as the minimum-zone deviation. Fig. 1 shows a profile defined by (16), as well as circumscribed and inscribed circles with their centres in the centre of the mean and the minimum-zone circles.

4 RESULTS FOR OTHER TYPES OF FORM ERRORS

Other types of form errors can be analysed in a similar way, but due to space limitation here, only the final results will be presented. As before, the analysis was carried out with the assumption that the form deviation is much smaller than the dimension of the real profile.

Theorem 2: *Let ΔP_{MZ} and ΔP_{LS} stand for the minimum-zone deviation and the least-squares deviation respectively. For various types of form errors the following inequities are obtained*

$$\Delta P_{LS} \le \left(1 + \frac{4(1-\cos\alpha)\sin\alpha}{2\alpha - \sin 2\alpha}\right)\Delta P_{MZ}$$ *roundness of a profile fragment with a central angle 2α with $\alpha \le \pi/2$.*

$$\Delta P_{LS} \le \frac{5}{2}\Delta P_{MZ}$$ *sphereness,*

$$\Delta P_{LS} \le \frac{5}{2}\Delta P_{MZ}$$ *2D straightness,*

$$\Delta P_{LS} \le \frac{8}{3}\Delta P_{MZ}$$ *3D straightness (axis straightness),*

$$\Delta P_{LS} \le 3\Delta P_{MZ}$$ *flatness of surface in the form of a rectangle,*

$$\Delta P_{LS} \le \left(1 + \frac{20}{3\pi}\right)\Delta P_{MZ}$$ *cylindricity.*

The above estimations are the best possible in the sense that the number next to ΔP_{MZ} on the right side of all inequities cannot be replaced by a smaller one.

Figs. 2a and 2b show the worst-case 3D straightness and cylindricity profiles used in the proof of the results of the theorem.

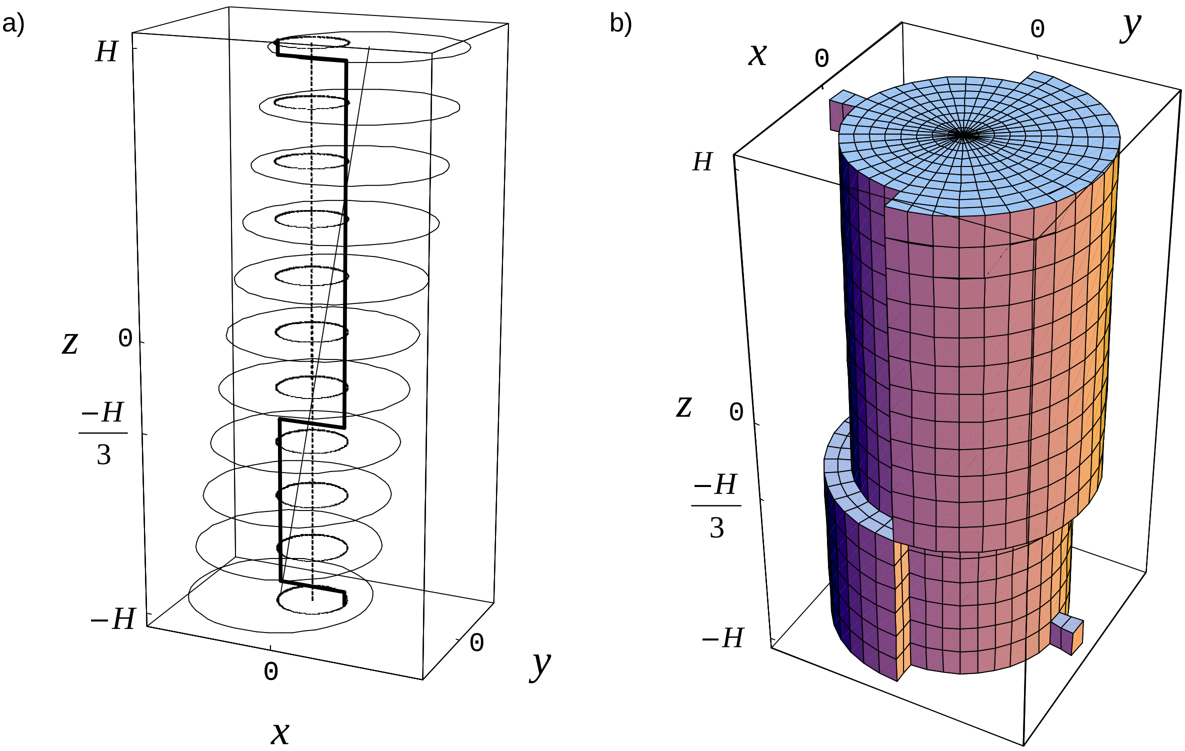

Figure 2. Worst-case (a) 3D straightness and (b) cylindricity profiles used in the proof of the results of theorem.

5 ANALYSIS OF INTERRUPTED PROFILES

An interrupted profile is a profile consisting of several fragments of continuous profiles. Interrupted profiles occur, for example, during measurement of roundness of toothed wheels or when we want to omit certain fragments of the measured profile, for instance, due to the occurrence of considerable deviations resulting from accidental scratches or dirt.

The analysis of interrupted profiles is actually more difficult. Below, a simple example of an interrupted 2D-straightness profile will be only considered. It will show that, for interrupted profiles, the ratio $\Delta P_{LS} / \Delta P_{MZ}$ can be even greater than in the case of continuous profiles.

Assuming that the straightness profile $P(x)$ is defined on the set $\Theta = [-l,-x_2] \cup [-x_1,x_1] \cup [x_2,l]$, $\Delta P_{LS} / \Delta P_{MZ}$ reaches the highest value when

$$P(x) = \begin{cases} P_{\max} \text{ when } x \in (-l,-l+e) \cup (0,x_1) \cup (x_2, l-e) \\ P_{\min} \text{ otherwise} \end{cases} . \quad (18)$$

Then, with $e \to 0$, we obtain:

$$k = 1 + \frac{3l(l^2 + x_1^2 - x_2^2)}{2(l^3 + x_1^3 - x_2^3)} . \quad (19)$$

For $x_1 = x_2$, we get $k = 5/2$, as given in theorem 2. Fig. 7 shows a diagram of the coefficient k as a function of x_1 for $l = 1$ and x_2 equal to 0.9, 0.99 and 0.999 respectively. One can see that the value of the ratio k for interrupted profiles can be greater than for continuous ones. Also, with appropriate x_1 and x_2, k can reach any high values, although such profiles are of no practical importance.

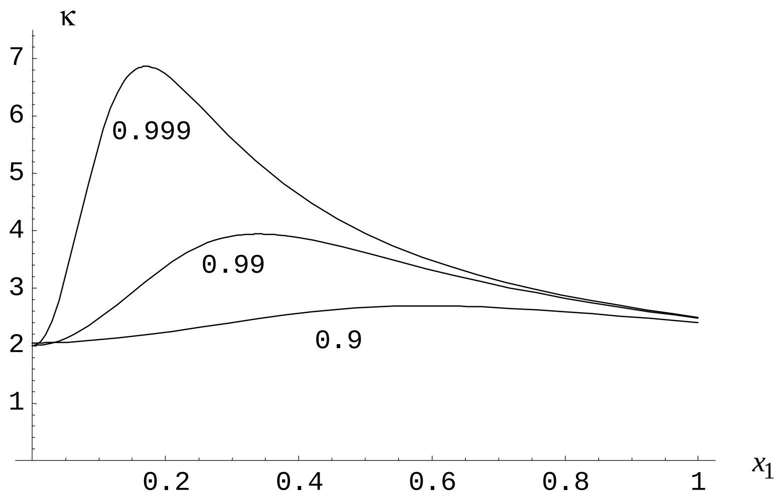

Figure 4. Diagram of the coefficient k as a function x_1 for $l = 1$ and various values of x_2.

6 ANALYSIS OF FILTERED PROFILES

The analysis of form profiles is often carried out for profiles filtered through low-pass filters, which enables separating the surface form from the surface waviness and roughness. The filtration reject rapid local changes of the profile, like those shown in fig. 1, for $j = 0$ and $j = p$. One should expect, therefore, that, for filtered profiles, the differences between the least square deviation and the minimum-zone deviation will be much smaller. In this Section, by means of the numerical analysis, we give a lower evaluation of the ratio k for filtered roundness profiles.

Let *N* be a given natural number and let us consider the class of roundness profiles of the form

$$P_N(j) = \sum_{i=2}^{N} a_i \cos ij + b_i \sin ij . \quad (20)$$

The profile $P_N(j)$ can be obtained from the measured profile by filtering (rejecting) the harmonics with numbers higher than N. For the given N, we shall determine the lower estimate $\bar{k}$ of the value of the ratio κ by means of the following procedure. We shall consider the profile $P(j,e)$ of the form (16). Let $P_N(j,e)$ stand for a profile obtained by rejecting harmonics with numbers higher than N from the profile $P(j,e)$. Let us define

$$\bar{k} = \max_{e \in (0,p/2)} \Delta P_{LS} / \Delta P_{MZ} , \quad (21)$$

where ΔP_{MZ} and ΔP_{LS} stand for the minimum-zone deviation and the least-squares deviation respectively of the filtered profile $P_N(j,e)$. Obviously, $\bar{k} \le k$ and $\bar{k}$ converges to the value $(1 + 4/p)$ determined in Theorem 1 when $N \to \infty$. Fig. 3 shows a diagram of the relationship between $\bar{k}$ and N obtained by means of numerical computations. Examples of the values of the obtained estimation $\bar{k}$ for various values of N are as follows: $\bar{k} = 1.79$ for $N = 15$, $\bar{k} = 2.00$ for $N = 50$. One can see that for $N = 50$ the obtained estimation $\bar{k}$ does not differ very much from the value of k for non-filtered profiles.

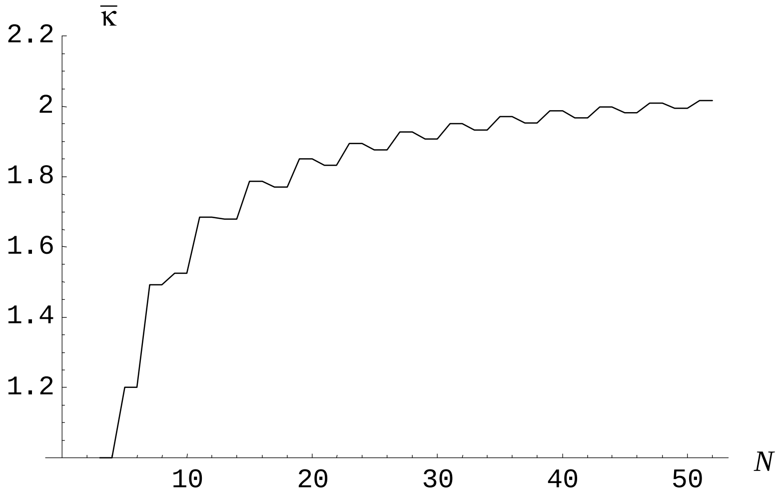

Figure 3. Diagram of the lower estimation of the worst-case value of $\Delta P_{LS} / \Delta P_{MZ}$ for filtered profiles

7 CONCLUSIONS

From the results of the theorem it is clear that the profile deviation determined in relation to the least squares profile can be much greater than the minimum-zone deviation. Since for cylindricity profiles, for instance, the ratio of the two deviations can be even greater then three, the application of the least-squares reference profiles may be sometimes questionable. For certain forms the determined deviations differ significantly from the intuitive notion of a form error. Also, it should be mentioned that the employment of modern computer equipment has resulted in an increase in the computation speed and so the advantage of the use of least-squares deviation characterised by simplicity loses its significance.

It should be noted that the analysis provided here is a theoretical one and shows only the greatest possible difference that may occur with various definitions of the deviations without investigating whether the considered profiles are reasonable from the practical point of view. It might be important, however, to study how big differences we should expect in practice, for example, what the distribution of ratio $\Delta P_{LS} / \Delta P_{MZ}$ is for certain objects made in a certain technological regime. Interesting though the problem is, it has not been addressed in this paper because a number of statistical tests would have to be carried out to provide details. Certain partial results on the problem are discussed in Ref. [3].

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AUTHORS: D. JANECKI and S. ADAMCZAK. Kielce University of Technology, Poland, Al. 1000-lecia PP 3, PL. 25-314 Kielce, Poland, tel. (0-4841) 34-24-534, fax (0-4841) 34-42-997
E-mail: adamczak@sabat.tu.kielce.pl and djanecki@ eden.tu.kielce.pl