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FREQUENCY ESTIMATION BY IDFT AND QUANTIZATION NOISE

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Abstract: Possibilities of error reductions of the frequency estimation by the multipoint interpolated discrete Fourier transform (IDFT) for rectangular and von Hann windows are described. An analysis is made to study the influence of the time-domain noise on the results of a DFT and interpolations. The quantization noise influence on the amplitude coefficients of DFT is in first approximation modeled by white noise in the time domain and Gaussian noise in the frequency domain. The noise influences on estimations are increased with the root of a number of the interpolation points, that is with apparent effective narrowing of the tapered window. Interpolations with larger number of points decrease the systematic errors and increase the noise distortion of results, with smaller number of points there is an inverse appearance. Increasing a number of the used DFT coefficients is reasonable until the systematic error drops under the noise floor.
Keywords: frequency estimation, IDFT, quantization noise

1 INTRODUCTION

To use the results of different DFT estimation algorithms for signal parameters in an efficient way [1], [2], [3], [4], it is necessary to have an idea of the uncertainty on the Fourier coefficients due to the noise disturbance. In most applications of parameters' estimations we are encountered with the quantization noise causing by digitalization during the measurement process. Since the amplitude coefficients of DFT are used in interpolations, we must first estimate the influence of the time domain noise $x(k)$ on the amplitude coefficients' distortions and then joint influence on the relative frequency estimation.

A sampled bandlimited analog signal $g(t)$, which is composed by a limited number of frequency components ($f_{m=0} = 0\text{Hz}$ - DC component), can be written as follows:

$$g(k\Delta t)_N = w(k)\,g(k\Delta t)_\infty = \sum_{m=0}^{M} A_m \sin(2p f_m k\,\Delta t + j_m) \tag{1}$$

We take from the infinite line of sampling points only a finite number N $(k = 0,1,\ldots,N-1)$ with a multiplication of the theoretically infinite sampled signal with a window of the finite duration $w(k)$. A_m, f_m and j_m are the amplitude, frequency and phase relevant to the measured component of the signal. Sampling frequency $f_S = 1/\Delta t$ is supposed to fulfil the Shannon theorem $f_S > 2 f_M$ (f_M - the highest frequency component). The DFT of signal on N sampled points at the spectral line i is given by:

$$G(i) = -\frac{j}{2} \sum_{m=0}^{M} A_m \left[W(i - q_m)e^{jj_m} - W(i + q_m)e^{-jj_m}\right] \tag{2}$$

q_m is the relative component frequency to base frequency resolution depending on the window span $\Delta f = 1/N\,\Delta t$ and can be written in two parts:

$$q_m = \frac{f_m}{\Delta f} = i_m + d_m \qquad -0.5 < d_m \le 0.5\ , \tag{3}$$

where i_m being an integer value. The displacement term d_m is caused by incoherent sampling. The local maximum in the amplitude part of DFT with the largest coefficients $|G(i)|$ and $|G(i+1)|$, surrounding the position of component m, must be found. Considering equations (2) and (3) we can write as follows:

$$|G(i_m)| = \frac{A_m}{2}|W(d_m)| \;, \quad |G(i_m \pm 1)| = \frac{A_m}{2}|W(1-d_m)| \tag{4}$$

The unknown amplitude A_m can be easily eliminated by ratio of coefficients.

$${}_2a_m = \frac{|G(i_m \pm 1)|}{|G(i_m)|} = \frac{|W(1-d_m)|}{|W(d_m)|} \tag{5}$$

2 FREQUENCY ESTIMATION

If the function $W(q)$ of window used is analytically known, the $d_m = f(a_m)$ can be expressed. For a rectangle [2] and Hann window [3] the following equations are valid:

$$|W_R(d_m)| \cong \frac{|\sin(pd_m)|}{pd_m / N} \qquad {}_2a_{mR} = \frac{d_m}{1-d_m} \qquad {}_2d_{mR} = \frac{{}_2a_{mR}}{1+{}_2a_{mR}} \tag{6}$$

$$|W_H(d_m)| \cong N\frac{|\sin(pd_m)|}{pd_m(1-d_m^2)} \qquad {}_2a_{mH} = \frac{1+d_m}{2-d_m} \qquad {}_2d_{mH} = 2\frac{{}_2a_{mH}-1}{{}_2a_{mH}+1} \tag{7}$$

The amplitude coefficients surrounding the component (Fig. 1) are composed of a short-range leakage contribution of the window spectrum weighted by the amplitude of component (the first item in (8)) and long-range leakage contributions (the second item in (8)):

$$|G(i_m)| = \left|-\frac{j}{2}A_m W(d_m)e^{ij_m} + \frac{j}{2}A_m W(2i_m + d_m)e^{-ij_m}\right| = \frac{A_m}{2}|W(d_m)| \pm |\Delta_W(i_m)| \tag{8}$$

The q_m being large enough the long-range leakage influences can be equalized in approximation $|\Delta(i_m-1)| \approx |\Delta(i_m)| \approx |\Delta(i_m+1)|$ and with subtraction eliminated by pairs [5]. In this way the threepoint estimation of a ratio can be written as:

$${}_3a_m = \frac{|G(i_m)| + |G(i_m-1)|}{|G(i_m)| + |G(i_m+1)|} \approx \frac{|W(d_m)| + |W(1+d_m)|}{|W(d_m)| + |W(1-d_m)|} \tag{9}$$

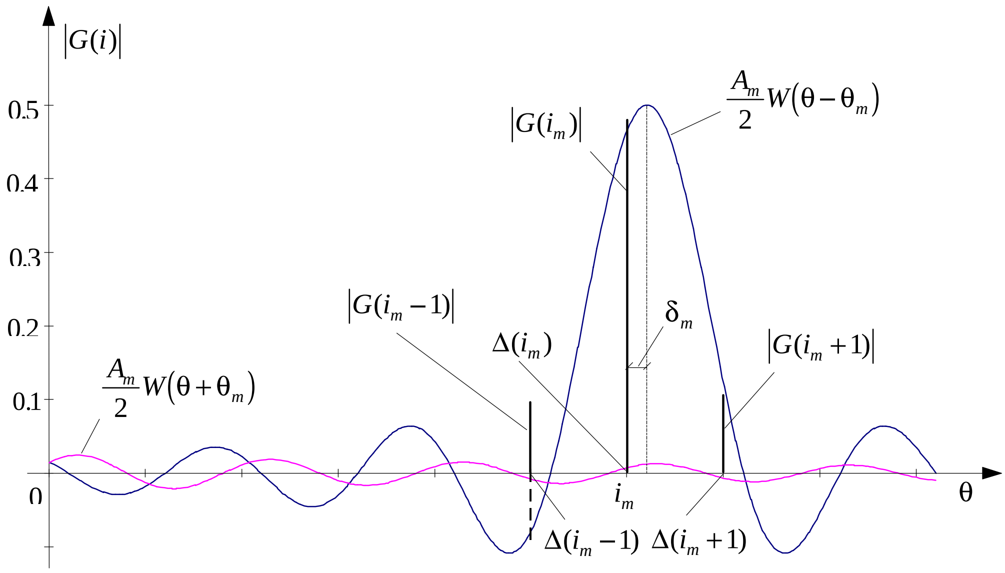

Figure 1. Leakage influence of the negative part of the sinus spectrum with rectangular window at the positive part of the spectrum ($i_m = 6$).

For the rectangular window (Fig. 1) the equation (9) can be rearranged in the following form:

$${}_3d_{mR} \approx \frac{1-{}_3a_{mR}}{1+{}_3a_{mR}} = s_d\,\frac{(|G(i_m+1)| + |G(i_m-1)|)}{2|G(i_m)| + \big||G(i_m+1)| - |G(i_m-1)|\big|} \tag{10}$$

The difference of the coefficients surrounding the largest one $|G(i_m)|$, gives us the sign of displacement d_m ($s_d = \mathrm{sign}(d_m) = \mathrm{sign}(|G(i_m+1)| - |G(i_m-1)|)$).

When Hann window is used, the main lobe is extended to four frequency resolution intervals (bins) $4\Delta f$ and all three coefficients of the maximum have the same sign.

$$ {}_3 d_{m\mathrm{H}} \approx 2\frac{1 - {}_3 a_{m\mathrm{H}}}{1 + {}_3 a_{m\mathrm{H}}} = 2\frac{|G(i_m+1)| - |G(i_m-1)|}{|G(i_m-1)| + 2|G(i_m)| + |G(i_m+1)|} \tag{11} $$

The displacement estimations with multipoint ($2r+1 = 3,5,7,\dots$) interpolations of DFT using Hann window can be written as follows:

$$ {}_{2r+1} d_{m\mathrm{H}} = (r+1)\cdot\frac{K_{n_1}\left[|G(i_m+1)| - |G(i_m-1)|\right] + \ldots + (-1)^l s_d K_{n_l}\left[|G(i_m+l)| + |G(i_m-l)|\right]}{K_{d_1}|G(i_m)| + K_{d_2}\left[|G(i_m+1)| + |G(i_m-1)|\right] + \ldots + (-1)^l s_d K_{d_l}\left[|G(i_m+l)| - |G(i_m-l)|\right]} \tag{12} $$

where r ($r = 1, 2, \dots$) is a number of the one side coefficients and l ($l \le r$) is the current index. The first term of the numerator in (12) is a difference of the side coefficients around the largest one and the rest of terms are sums of symmetrical pairs of coefficients ($|G(i_m \pm l)|$). All terms $[\cdot]$ are weighted by $K_{n_l} = \binom{2(r-1)}{(r-l)} - \binom{2(r-1)}{(r-l-2)}$. The indefinite states $\binom{x_1}{-x_2}$ are set to zero and when r is 1, it is consider $\binom{0}{0} = 1$. The signs of weights alter successively. The largest coefficient in the denominator is weighted by $K_{d_1} = \binom{2r}{r}$, the sum of first side coefficients by $K_{d_2} = \binom{2r}{(r-1)}$, and the differences of symmetrically located coefficients are weighted by weights $K_{d_{l+1}} = \binom{2r}{(r-l)}$. The signs of weights also alter successively $(-1)^l$.

3 QUANTIZATION NOISE

The time domain noise $x(k)$ is in the first approximation modeled to be white with uniform distribution from $-x_{\max}$ to $+x_{\max}$ [6]. The mean of noise is $u(x) \to 0$ and standard deviation is $s_x = x(k)_{\max}/\sqrt{3}$. Since it is added to the signal $g(k) + x(k)$, it goes through the same mathematical DFT procedure as signal $g(k)$ ($\sum \begin{bmatrix}\sin \\ \text{or} \\ \cos\end{bmatrix}(2pik/N)\cdot w(k)\cdot x(k)$). It is seen that the noise on the Fourier coefficients is derived from a sum of weighted random variables [7]. The standard deviation of the amplitude DFT coefficients ($s_{\mathrm{real}} = s_{\mathrm{imag}} = s_{|\mathrm{DFT}|}$ [8]) can be expressed as:

$$ s^2_{|\mathrm{DFT}|} = \sum_{k=0}^{N-1}\begin{bmatrix}\sin \\ \text{or} \\ \cos\end{bmatrix}(2pik/N)\cdot w(k)\cdot s_x\Bigg]^2 = s_x^2\,\frac{1}{2}\sum_{k=0}^{N-1} w^2(k) \tag{13} $$

Considering Parseval's theorem for the DFT $1/N \cdot \sum |g(k)|^2 = \sum |G(i)|^2$ [9] the standard deviation has to be reduced by N.

$$ s_{|\mathrm{DFT}|} = s_x\,\frac{1}{N\sqrt{2}}\sqrt{\sum_{k=0}^{N-1} w^2(k)} \tag{14} $$

The "absolute" form of the standard deviation (14) is usually related to the values of DFT coefficients of interest. In coherent sampling, the largest local amplitude DFT coefficient is equal $|G(i_m)| = A_m/2N \cdot \sum w(k)$ and the relative form of the standard deviation can be written as:

$$ s^*_{|\mathrm{DFT}|} = \frac{s_{|\mathrm{DFT}|}}{|G(i_m)|} = \frac{s_x}{A_m}\sqrt{2}\,\frac{\sqrt{\sum_{k=0}^{N-1} w^2(k)}}{\sum_{k=0}^{N-1} w(k)} = \frac{s_x}{A_m}\frac{\sqrt{2}}{\sqrt{N}}\sqrt{ENBW} \tag{15} $$

The relative standard deviation of the DFT depends on: the amplitude value of component relative to the noise level s_x, a number of the sampling points and the used window. The root of the equivalent noise bandwidth *ENBW* [9] is a factor determining the size of the standard deviation when using different windows. The least distortion of DFT coefficients is with the rectangular window ($ENBW > ENBW_{\mathrm{rect.}} = 1$). The equation (15) is valid for amplitude coefficients of DFT that are sufficiently moved away from margins of spectral field ($q = 0, N/2$ - influences of the short-range leakage!).

In the measurement practice we are encountered with incoherent sampling ($d_m \neq 0$), where the frequency interpolation can be used. Distortions of DFT coefficients and their number in interpolation have significant influence on the uncertainty of displacement d_m estimation and indirectly the uncertainty of frequency q_m. Distributions of errors ($E = |G(i)|_{\text{noise}} - |G(i)|_{\text{noiseless}}$) of the largest amplitude coefficients have very similar (Gaussian) shapes with almost equal standard deviations $s_{|\text{DFT}|}(i_m - r) \approx s_{|\text{DFT}|}(i_m) \approx s_{|\text{DFT}|}(i_m + r) = s_{|\text{DFT}|}$ (14), if the time domain noise is statisticaly independent of signal and sampling process (suitable high sampling frequency $f_S / f_M >> 2$ and a number of bits of the A/D converter is high $n \geq 8$). The standard deviation of the displacement d_M, as depending quantity, can generally be expressed as [10]:

$$s_{d_m}^2 = \sum_{p=i_m-r}^{p=i_m+r} \left(c(p) \cdot s_{|G(p)|}\right)^2 + 2 \sum_{p=i_m-r}^{p=i_m+r-1} \sum_{v=p+1}^{v=i_m+r} \left[r\left(|G(p)|, |G(v)|\right) \cdot c(p) \cdot c(v) \cdot s_{|G(p)|} \cdot s_{|G(v)|} \right], \tag{16}$$

where $c(p) = \partial d_m / \partial |G(p)|$ ($p = i_m - r, \ldots, i_m + r$) is the sensitivity coefficient associated with the amplitude coefficient $|G(p)|$, and $r(|G(p)|, |G(v)|)$ ($p \neq v$) is the correlation coefficient, the measure of the relative mutual dependence of two DFT amplitude coefficients. In the case of Hann window, each two successive amplitude coefficients have correlation factor $r(|G(p)|, |G(p+1)|) = 2/3$, and the amplitude coefficients with current index two apart, have $r(|G(p)|, |G(p+2)|) = 1/6$. Other correlation coefficients are zero.

Since the standard deviations of the amplitude coefficients are almost equal $s_{|G(p)|} \cong s_{|G(v)|} = s_{|\text{DFT}|}$, it is possible to formulate exspression for the standard deviation of displacement. For the threepoint interpolation using Hann window (11), it can be expressed as:

$${}_3s_{d_m}^2 = s_{|\text{DFT}|}^2 \left[\left[c^2(i_{m-1}) + c^2(i_m) + c^2(i_{m+1}) \right] + \frac{4}{3}\left[c(i_m - 1) \cdot c(i_m) + c(i_m) \cdot c(i_m + 1) \right] + \frac{2}{6}\left[c(i_m - 1) \cdot c(i_m + 1) \right] \right] \tag{17}$$

The same mathematical procedure can be used for other higher multipoint interpolations (12).

$${}_{2r+1}s_{d_m} = s_{|\text{DFT}|} \sqrt{\sum_{p=i_m-r}^{p=i_m+r} c^2(p) + \frac{4}{3} \cdot \sum_{p=i_m-r}^{p=i_m+r-1} c(p) \cdot c(p+1) + \frac{1}{3} \cdot \sum_{p=i_m-r}^{p=i_m+r-2} c(p) \cdot c(p+2)} \tag{18}$$

Sensitivity coefficients $c(p)$ have forms like $c(p) = \left[(d_{2r+1})' \cdot n_{2r+1} - (n_{2r+1})' \cdot d_{2r+1} \right] / n_{2r+1}^2$, where d_{2r+1}, n_{2r+1} are the denominator and the nominator of fraction (12), respectively. Both of them are sums of the weighted amplitude DFT coefficients, which are changing with the relative displacement (the short-range leakage influences). For this reason the standard deviations of displacement change their values periodically $-0.5 < d_m \leq 0.5$ (Fig. 2).

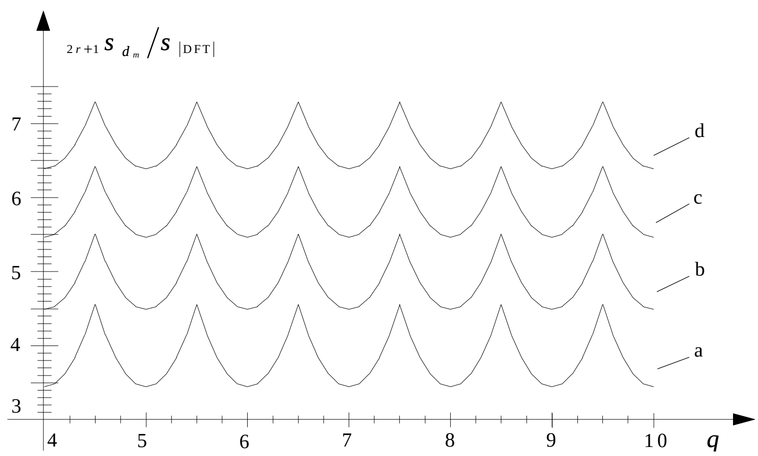

Figure 2. Standard deviations of the displacement estimation related to the standard deviation of the amplitude coefficient (a: $2r + 1 = 3$, b: $2r + 1 = 5$, c: $2r + 1 = 7$, d: $2r + 1 = 9$)

Errors of the relative frequency estimations with different numbers of interpolation points also have normal distribution. The standard deviation of the threepoint frequency estimation is about 4 times higher than the standard deviation of the amplitude DFT coefficient (Fig. 2:a). The lowest value (${}_3 s_{d_m} / s_{|DFT|} = 3.44$) is attained at the integer values of relative frequency ($d_m \approx 0$) and the highest ratio (${}_3 s_{d_m} / s_{|DFT|} = 4.56$) is at the worst cases of incoherent sampling ($d_m \approx 0.5$). The standard deviations of other higher multipoint interpolations in comparison to the basic threepoint increase as follows:

$$ {}_3 s_{d_{mH}} / {}_5 s_{d_{mH}} / {}_7 s_{d_{mH}} / {}_9 s_{d_{mH}} \approx 1/1.27/1.52/1.76 \qquad (19)$$

As an approximate value of increasing of the standard deviations we can make a rough calculation with numbers of the interpolation points ${}_{2r+1} d_{d_{mH}} / {}_3 d_{d_{mH}} \approx \sqrt{(2r+1)/3}$.

The results of simulation, where the relative frequency has been changed, show that the noise influences on estimations (s_{E_q} , $|E_q|_{max}$) are increased with the root of a number of the interpolation points, that is with apparent effective narrowing of the tapered window $\Delta T_{w(k)}$ ($ENBW > 1$). At the same time the systematic errors decrease with the increasing number of points. The increasing of a number of the used DFT coefficients is reasonable until the systematic error drops under the noise error. After this point with increasing of the relative frequency q_m (a number of swings of the measured signal in the time interval or with spacing between two frequency components ($\propto 2 \cdot q_m$), it is logical to decrease the number of interpolation points.

Figure 3. The use of multipoint DFT interpolations for 10-bit A/D converter.

The borders of the relative frequency, where one interpolation can pass over another, depend on a number n of bits of the A/D converter. If the model of an ideal A/D converter ($\Delta = G_R / (2^n - 1) = 2A_m / (2^n - 1)$; G_R - full-scale range) is used in simulation and the parameters of sinus function are changed, we get the errors as presented on Fig. 3 and 4. With 10 bit A/D converter (Fig. 3), which is very frequently used in industrial environment, it is convenient to use the threepoint DFT interpolation with Hann window in the interval $1 \le q_m < 2$ and from $6.5 \le q_m$ onward. Between values 2 and 6.5 of the relative frequency it is better to use fivepoint interpolation (the lower thick line on Fig. 3). Multipoint interpolations ($2r + 1 \ge 7$) present worse results: at lower q_m owing to the systematic error ("window width"), at higher values of q_m owing to larger noise sensibility. The 16-bit A/D converter (Fig. 4) enables the use of the sevenpoint DFT interpolation in the interval $3.5 \le q_m < 8$.

Figure 4. The use of multipoint DFT interpolations for 16-bit A/D converter.

4 SUMMARY

Parameter's estimations of periodical signals are more and more frequently performed in frequency domain. In the paper we have pointed out the possibility of error reduction of the frequency estimation by the multipoint interpolated discrete Fourier transform (DFT) for rectangular and von Hann windows. The noise influences on estimations are increased with the root of a number of the interpolation points, that is with apparent effective narrowing of the tapered window. Interpolations with larger number of points decrease the systematic errors and increase the noise distortion of results, with smaller number of points there is an inverse appearance.

The use of a suitable interpolation algorithm depends on the effective bits of the A/D conversion, on the position of the frequency component of signal and on mutual components interspacing along the frequency axis. The borders of the relative frequency where one interpolation can pass over another depend on the number n of bits of the A/D converter. If we use different optimal interpolation algorithms for frequencies' estimations of multicomponent signal, we change adaptively the apparent window shape for the particular component.

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