XVI IMEKO World Congress
Measurement - Supports Science - Improves Technology - Protects Environment ... and Provides Employment - Now and in the Future
Vienna, AUSTRIA, 2000, September 25-28

CORRELATION EFFECT ON EFFECTIVE DEGREES OF FREEDOM

V.Y. Aibe
Fluids Laboratory
Division of Mechanical Metrology
National Institute of Metrology Standardization and Industrial Quality, RJ, Brazil

*Abstract: The mathematical model of the measurement in the "Guide to the Expression of Uncertainty in Measurement" is Y=f(X_1, X_2,...,X_N), where **Y** is the meansurand and X_1, X_2,...,X_N are the influence quantities. The expanded uncertainty U is obtained by U = k u_c(y), where u_c(y) is the combined uncertainty and **k** is the coverage factor that depends on the effective degrees of freedom.*
The value of effective degrees of freedom is obtained from the Welch-Satterthwaite formula. It is assumed that there isn't correlation between influence quantities.
In this paper will be presented an example in which the use of this formula would be inadequate because there is a strong correlation between influence quantities. An alternative model of measurement was used to compare with the first model in the example case. Although the combined uncertainty of the two models is almost the same, there are differences between the expanded uncertainty, because there is a significant difference in the respective values of two effective degrees of freedom. This difference is caused by strong correlation between influence quantities.
Keywords: manuscript, IMEKO, World Congress, Estimation of Uncertainty and Error in Measurement, effective degrees of freedom,

1 INTRODUCTION

MODELING OF MEASUREMENT AND UNCERTAINTY

The mathematical model of the measurement in the "Guide to the Expression of Uncertainty in Measurement" [1] is Y=f(X_1, X_2,...,X_N), where Y is the meansurand and X_1, X_2,...X_N are the influence quantities. Let y the estimator of quantity Y and x_i the estimator of X_i, , $y=f(x_1, x_2,...,x_N)$. The expanded uncertainty U is obtained by $U = k\, u_c(y)$, where $u_c(y)$ is the combined uncertainty and k is the coverage factor that depends on the effective degrees of freedom and level of confidence.

The combined uncertainty is calculated by following formula:

$$u_c^2(y) = \sum_{i=1}^{N} \left[\frac{\partial f}{\partial x_i} \right]^2 u^2(x_i) + 2 \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} u(x_i, x_j) \qquad (1)$$

Where:
$u(x_i)$ is the standard uncertainty of quantity X_i
$u(x_i\,,x_j)$ is the estimated covariance associated with quantities X_i and X_j

$$\frac{\partial f}{\partial x_i} = \frac{\partial f(x_i)}{\partial X_i} = c_i \quad \text{is the sensivity coefficient of quantity } X_i \qquad (2)$$

The value of the effective degrees of freedom is obtained from the Welch-Satterthwaite formula:

$$g_{eff} = \frac{u_c^4(y)}{\sum_{i=1}^{N} \frac{\left(c_i \cdot u(x_i)\right)^4}{n_i}} \qquad (3)$$

This formula assumed that there is not correlation between influence quantities [2]. In this paper will be presented an example in which the use of this formula can be inadequate because there is a strong correlation between influence quantities. An alternative model of measurement was used to be compare with the first model.

The value of k is obtained from Student distribution table and g_{eff}, ,normally the adopted level of confidence is 95,45%.

In some cases the estimate *y* can be obtained using following model:

$$y = f\left(\frac{1}{n}\sum_{k=1}^{n} X_{1,k}, \frac{1}{n}\sum_{k=1}^{n} X_{2,k}, \ldots, \frac{1}{n}\sum_{k=1}^{n} X_{N,k}\right) = f(\overline{X}_1, \overline{X}_2, \ldots, \overline{X}_N) \quad (4)$$

Where *n* is the number of observations of each quantity and $X_{i,k}$ is the observation *k* of quantity X_i. . N is the number of quantities. In this model the correlation coefficients between the influence quantities can the obtained by their observed values.

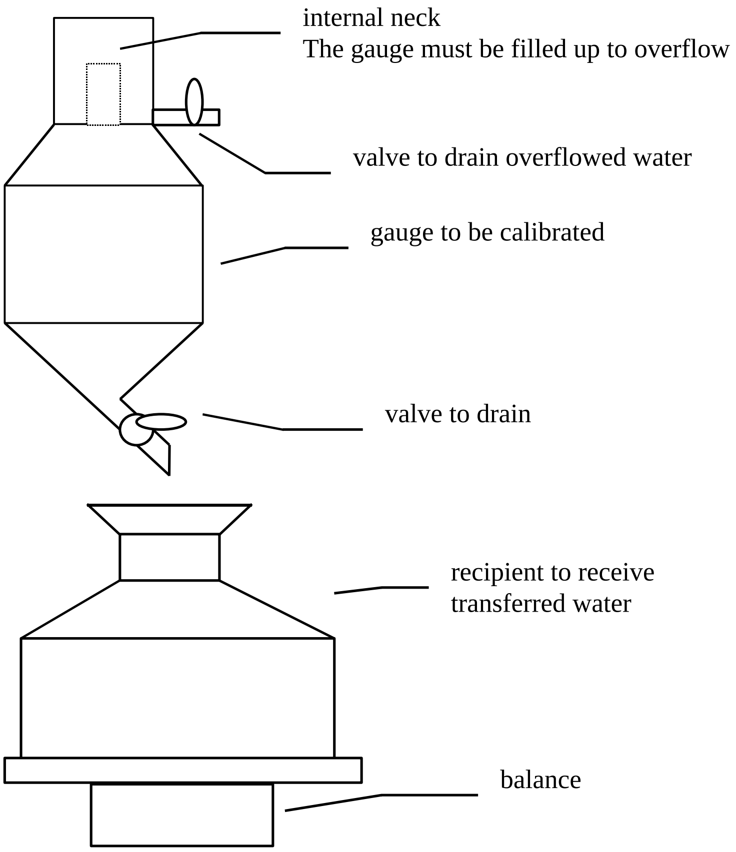
The alternative model is: $y = \overline{Y} = \frac{1}{n}\sum_{k=1}^{n} Y_k = \sum_{k=1}^{n} f(X_{1,k}, X_{2,k}, \ldots\ldots, X_{N,k})$ (5)

In this case, if each measuring of *Y* is independent, the correlation coefficients can be ignored.

Although the combined uncertainty of two models is almost the same, there are differences between the expanded uncertainty value, because there is a big difference between the respective value of effective degrees of freedom. This difference is caused by strong correlation between influence quantities.

2 GAUGE CALIBRATION EXAMPLE

There is a gauge to measure the volume of liquid that is transferred to a recipient or that is contained in the gauge. Here will be considered the transferred volume. In this case the calibration by the weighing method the transferred liquid is collected in another recipient. The mass and density of liquid are determined to calculate its volume.



2.1 Mathematical model

Normally water is used in the calibration. The volume is calculated by the following formula:

$$V = \left(\frac{M_w}{\rho_w - \rho_{air}}\right) \times \left(1 - \frac{\rho_{air}}{\rho_{weight}}\right) \times \left(1 - \gamma(T_w - T_r)\right) \qquad (6)$$

Where:

V= transferred water volume at reference temperature
M_w = transferred water apparent mass
r_w = water density during calibration
r_{air} = air density
r_{weight} = weight density of weight that was used to calibrate the balance
T_w = water temperature during calibration
T_r = reference temperature of gauge
γ = volumetric coefficient of expansion

The uncertainty of volume determination in each transference will be considered instead of uncertainty of average volume, so the standard deviation will not be divided by square root of number of repeated observations.

2.2 Results of measurements

Ten measurements of transferred volume were made and below are the results:

Table 1: Results of measurements

	Apparent mass of transferred water M_w (kg)	Water temperature T_w (^{o}C)	Water density r_w (kg/m^3)	Transferred water volume V (m^3)
1	527.369	19.80	998.39	0.5287808
2	527.375	19.85	998,37	0.5287962
3	527.349	19.92	998.36	0.5287736
4	527.382	19.85	998.37	0.5288032
5	527.367	19.85	998.37	0.5287882
6	527.367	19.90	998.37	0.5287869
7	527.387	19.95	998.35	0.5288163
8	527.287	20.68	998.20	0.5287760
9	527.287	20.67	998.20	0.5287762
10	527.281	20.76	998.18	0.5287796
average	527.345	20.123	998.316	0.5287877
stand. dev.	0.0427	0.4033	0.,0854	0.00001378

Table 2: Estimated correlation coefficient matrix: $r(x_i,x_j)$

	M_w	r_w	T_w
M_w	1	0.693347	-0.837770
r_w	0.693347	1	-0.737475
T_w	-0.837770	-0.737475	1

2.3 First model: $V = f(\overline{M}_w, \overline{\rho}_w, \rho_{aor}, \rho_{weight}, \gamma, \overline{T}_w, T_r)$

The correlation between M_w., r_w, and T_w are considered.

Table 3: Uncertainty budget of first model

Input quantities X_i	Estimated value x_i	Estimated expanded uncertainty	Unit	Sensivity coefficient c_i	Standard uncertainty $u(x_i)$	Contribution to uncertainty $u_i(y)$	Degrees of freedom d.f.
M_w.	527.345	0.0212	kg	0.00100275	0..0106066	1.06357E-05	∞
M_w. (*)			m³	0.00100275	0.0427	4.28268E-05	9
r_{ai}r	1.2	0.02	g/ml	0.00046421	0.01	4.64214 E-06	∞
r_w	998.32	0.10	g/ml	-0.0005303	5.7735E-02	-3.06182E-05	∞
r_{air} (*)				-0.0005303	8.480E-02	-4.4971 E-05	9
w_{eight}	8000	200	g/ml	9.9164E-09	115.47005	1,14504 E-06	∞
T_w	20.12	0.10	°C	-2.538E-05	0.05	-1.2691 E-06	∞
T_w (*)				-2.538E-05	0.4033	-1.02371E-05	9
γ.	0.000048	0.000008	1/°C	-0.1707994	4.6188E-06	-7.8889 E-07	∞

Volume	Expanded uncertainty	Combined uncertainty	Coverage factor. k	Degrees of freedom	Relative uncertainty
m³	m³	m³			%
0.52879	0.00050	0.000036	13.97	1.9	0.094

(*) repeatability component of contribution to uncertainty

2.4 Second model considering $V = \frac{1}{n}\sum_{i=1}^{n} V_i \qquad V_i = f(M_w, r_w, r_{air}, r_{weight}, g, t, t_r)$

Table 4: Uncertainty budget of first model

Input quantities X_i	Estimated value x_i	Estimated expanded uncertainty	Unit	Sensivity coefficient c_i	Standard uncertainty $u(x_i)$	Contribution to uncertainty $u_i(y)$	Degrees of freedom d.f.
M_w.	527.345	0.0212	kg	0.00100275	0.0106066	1.0636E-05	∞
V (*)			m³	1	0.00001378	1.3785 E-05	9
r_{air}	1.2	0.02	kg/m³	0.00046421	0.01	4.64214 E-06	∞
r_w	998.32	0.10	kg/m³	-0.0005303	0.057735	-3.0618 E-05	∞
r_{weight}	8000	200	kg/m³	9.9164 E-09	115.47005	1.14504 E-06	∞
T_w	20.12	0.10	°C	-2.5382E-05	0.05	-1.2691 E-06	∞
γ.	0.000048	0.000008	1/°C	-0.0650413	4.6188E-06	-3.0041 E-07	∞

Volume	Expanded uncertainty	Combined uncertainty	Coverage factor. k	Degrees of freedom	Relative uncertainty
m³	m³	m³			%
0.52879	0.000072	0.000036	2.0063	399	0.013

2.5- Upper and lower limit of quantities values

A simple numeric method was used to evaluate the limit of possible variation of transferred volume. The variation of each input quantity was estimated combining the standard uncertainty of each measurement and standard deviation.

Table 5: Estimated upper and lower limit of each input quantity

Input quantity X_i	Estimated values x_i	Estimated uncertainty U	Unit	upper limit $U_p(xi)$	Lower limit $L_w(xi)$
M_w	527,345	0.10	kg	527.4451	527.2451
r_{air}	1,20	0.02	kg/m^3	1.22	1.18
r_w	998.316	0.22	kg/m^3	998.0964	998.536
r_{weight}	8000	200	kg/m^3	8200	7.800
T_w	20.12	0.95	^{o}C	19.17	21.07
g	0.000048	0.000008	1/^{o}C	0.000040	0.00006

The upper limit and lower limit of volume were calculated by:

$$U_p(V)=\left(\frac{U_p(M_w)}{L_w(r_w)-U_p(r_{air})}\right)\times\left(1-\frac{U_p(r_{air})}{L_w(r_{weight})}\right)\times\left(1-L_w(\gamma)\cdot(L_w(T_w)-T_r)\right)$$

The upper limit and lower limit of volume were calculated by:

$$L_w(V)=\left(\frac{L_w(M_w)}{U_p(r_w)-L_w(r_{air})}\right)\times\left(1-\frac{L_w(r_{air})}{U_p(r_{weight})}\right)\times\left(1-U_P(\gamma)\cdot(U_P(T_w)-T_r)\right)$$

Where:

$U_p(X)$ is the upper limit of quantity X *and* $L_w(X)$ is the lower limit of quantity X

Table 6: Estimated upper and lower limit of each input quantity

	upper limit $U_p(V)$	Lower limit $L_w(V)$	range/2
transferred volume (m^3)	0.5290366	0.52853089	0.00025

The range of volume variation determined in this way is greater than the actual value, then the expanded uncertainty must be lower than 0,00025m^3.

3 CONCLUSIONS

The two models have the same combined uncertainties but the expanded uncertainty of the first model is almost seven times bigger then the second model, because there is a big difference between the effective degrees of freedom.

In the formula (1) of expanded uncertainty the correlation is considered. The formula (3) of Welch-Satterthwaite assumed that there is not correlation between influence quantities. Then for some cases the formula (3) can be inadequate.

The expanded uncertainty obtained from first model is 0.0005 m^3 and from second model is 0.000072m^3. In the condition where there is no uncertainty type **A** the expanded uncertainty is 0.000065m^3. The evaluation of range of volume variation showed that expanded uncertainty must be lower than 0.00025m^3 , therefore the result of second model is better.

REFERENCES

[1] ISO, Guide to the Expression of Uncertainty in Measurement, (1993), 9-24; 59-65.
[2] SATTERTHWAITE, F. E. (1941), Psychometrika 6, 309-316; (1946); (1946) Biometrics Bull. 2(6), 110-114.