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TWO-INTERVAL OVERLAP TEST FOR FAILURE PREDICTION IN KALMAN FILTER

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Abstract. An approach to failure prediction in Kalman filter is developed which uses confidence and tolerance intervals for a innovation sequence. The algorithm proposed is based on the criterion of overlapping these intervals and allows the detection of potential failures in Kalman filter. The problem of failure prediction in multidimensional Kalman filters is solved too. In this case a multidimensional innovation sequence is replaced by one- dimensional sequences.

Key words: Kalman filter, innovation sequence, failure prediction, confidence intervals, tolerance intervals.

1. INTRODUCTION

In the realization of algorithms of control and navigation in real time, functional diagnosis of a computer algorithm of Kalman filtration which is subject to failure and errors, is a topical problem for today. To solve the problem, different methods of control and diagnosis are used. [1-5].

In [1], the problem of control of Kalman filter has been solved using the Chi-square- criterion, and the solution uses statistical data on the distinction between Kalman filter estimates and time extrapolation. In [2,4,5], innovation sequences are used for which, if the system is functioning properly, the normalized innovation sequence in Kalman filter (which is in agreement with the dynamic model) is Gaussian white noise with zero mean and identity covariance matrix. An efficient algorithm for the control and diagnosis of Kalman filter has also been proposed in [3]. It is based on introduced analytical redundancy and allows the totality of controlling conditions (algebratic invariants) to be obtained, which are used to detect and localize defects.

The need for predicting potential failures arises in the operation of a certain class of dynamic systems. This prediction makes it possible to take a decision on the necessity of using controlling actions on the process of technical operation and servicing the system under study before actual failure occurs. Despite a wide variety of controlling techniques for Kalman filter, there are no methods for predicting its states. In this study failure prediction problem is solved on the basis of building and checking overlapping confidence and tolerance intervals for the innovation sequence.

2. MATHEMATICAL MODEL

Let us consider the problem of estimation of state coordinates of the lineer dynamic system

$$x_{i+1} = Ax_i + w_{i+1} \qquad (1)$$

using measured values

$$y_{i+1} = Cx_i + \mathrm{v}_{i+1}\,, \qquad (2)$$

where A and C are known parameters $(C \neq 0)$, $\{w_i\}$ is the sequence of independent identically distributed random values (system noise) with distribution $N(0, \sigma_w^2)$; $\{\mathrm{v}_i\}$ is the sequence (independent of w_i) of identically distributed random values (measuring noise) with distribution $N(0,\sigma_\mathrm{v}^2)$; and x_0 is the initial condition independent of $\{w_i\}$ and $\{\mathrm{v}_i\}$.

In the well studied Gaussian distribution of the x_0 value, a minimum of mean square deviation $E(x_i - \hat{x}_i)^2$ (E is a statistical averaging operator) of the $\hat{x}_i$ estimate from the x_i state represents a discrete lineer optimal Kalman filter

$$\hat{x}_{i+1} = A\hat{x}_I + \frac{AP_iC}{\sigma^2{}_V + C^2 P_i}\left(y_{i+1} - C\hat{x}_i\right), \hat{x}_0 = x_0, \qquad (3)$$

$$P_{i+1} = A^2 P_i + \sigma^2{}_w - \frac{(AP_iC)^2}{\sigma^2{}_v + C^2 P_i},$$

where P_i is the variance of estimate errors. Under normal operation of Kalman filter (3), the innovation sequence $v_i = y_{i+1} - C\hat{x}_i$ is the white Gaussian noise with zero mean and variance $\sigma_\mathrm{v}^2 + C^2P_i$, i.e. $\mathrm{v}_i \sim N(0,\ \sigma_\mathrm{v}^2 + C^2P_i)$ [6]. To realize technical state diagnosis, it is more convenient to use normalized innovation sequence of the form

$$\tilde{v}_i = \frac{v_i}{\sqrt{\sigma^2{}_v + C^2 P_i}} \qquad (4)$$

because $\tilde{v}_i \sim N(0,1)$ in this case.

3. BUILDING CONFIDENCE INTERVAL FOR $\tilde{\nu}_i$

Choosing the confidence probability β from the condition P={u≤u_β }=β , we can build the confidence interval for the normalized innovation sequence $\tilde{\nu}_i$ using standard normal distribution quantiles. Such an interval will be called a priori, since it is determined once before starting testing and remains constant during estimation process.

The confidence interval which is formed with respect to the normalized innovation sequence obtained by prosessing actual measurements, we call the a posteriori interval. To determine this interval, sampling characteristics can be used

$$\bar{\tilde{\nu}} = \frac{1}{M}\sum_{i=1}^{M}\tilde{\nu}_i \,; \qquad \hat{s}^2 = \frac{1}{M-1}\sum_{i=1}^{M}\left(\tilde{\nu}_i - \bar{\tilde{\nu}}\right)^2 \,, \tag{5}$$

where $\bar{\tilde{\nu}}$ and $\hat{s}$ are the sampling mean and sampling variance of $\tilde{\nu}_i$, respectively, and M is the number of realizations being used.

Simultaneous replacement of the mathematical expection and variance of the normalized innovation sequence by their sampling values $\bar{\tilde{\nu}}$ and $\hat{s}$ in the course of building a confidence interval causes us abandon the use of normal disribution quantiles. As a statistical parameter, in this case the standardized maximum deviation can be employed [7]

$$\tau_i = \frac{\tilde{\nu}_i - \bar{\tilde{\nu}}}{\hat{s}} \tag{6}$$

The value τ_i that expresses the deviation of value $\tilde{\nu}_i$ from the mean value $\bar{\tilde{\nu}}$ rated by the standard sampling deviation $\hat{s}$, has a special distribution which depends only on the volume of sampling M. This distribution is called τ–distribution. Its quantiles have been tabulated for different M and confidence probability β. Thus, using given β and M and condition P{$\tau_i \le \tau_\beta$} = β, the confidence interval for the normalized innovation sequence $\tilde{\nu}_i$ can be easily built using τ - distribution quantiles. If M→∞, statistics (6) tends to the standard normal distribution.

4. CRITERION OF OVERLAPPING OF PRIORI AND POSTERIORI CONFIDENCE INTERVALS

Analitical theory of failure detection has been proposed in [8,9]. The theory uses the criterion of overlapping two confidence regions- one was built with respect to the nominal trajectory of a system in which failures are absent, and other was built with respect to Kalman's filter estimates obtained in the course of processing actual measurements. The efficiency and numerical verification of this approach were studied in detail in the mentioned papers.

Using priori and posteriori confidence intervals built for the normalized innovation sequence and the overlapping condition proposed in [8] for two confidence regions, we can consider the following approach to the detection of failures. If the mentioned confidence intervals are covered, then true value $\tilde{\nu}_i$ can lie in any one of these intervals. In this connection we can conclude that Kalman filter is functioning properly. When these intervals are distinct, the estimation system state can be substantially different from the nominal state and failure should be fixed. Consequently, the problem of failure detection can be reduced to the search for a single point whose simultaneous presence in the both intervals is a necessary and sufficient condition for their overlapping, and absence is a necessary and sufficient condition for failure detection in Kalman filter.

In the diagnosis of an estimation system in real conditions, the proposed algorithm can be reduced to the following sequence of calculations.

Algorithm 1.

1. At the given confidence probability, build a priori confidence interval [a,b] (Fig.1) for the normalized innovation sequence $\tilde{\nu}_i$~N(0,1) using standard normal distribution quantiles.
2. Using (3) and (4) calculate Kalman's estimate of system state $\hat{x}_i$ and value $\tilde{\nu}_i$ of normalized innovation sequence at step i.
3. Calculate the values of sampling mean and variance by formulas

$$\bar{\tilde{\nu}}_i = \frac{1}{M}\sum_{j=i-M+1}^{i}\tilde{\nu}_j \,; \hat{s}_i^2 = \frac{1}{M-1}\sum_{j=i-M+1}^{i}\left(\tilde{\nu}_j - \bar{\tilde{\nu}}_i\right)^2 \,, \tag{7}$$

where M is the width of "sliding window", and using τ– distribution quantiles, determine the boundaries [c,d] of a posteriori confidence interval for $\tilde{\nu}_i$.

4. Check the overlapping condition for the confidence intervals obtained.
5. Repeat calculations for the next time instant beginning from step 2.

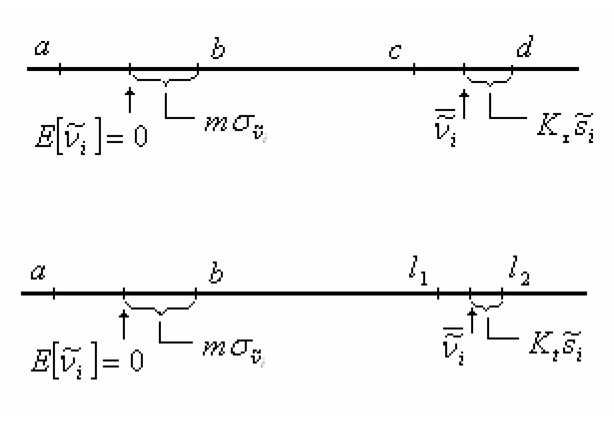

Fig. 1. Determination of confidence and tolerance intervals for the detection and prediction of failures

To obtain the overlapping condition for confidence intervals, we can use their geometric interpretation. It follows from Fig.1 that the mentioned intervals do not overlap if conditions

$$m\sigma_{\tilde{v}_i} + K_\tau \hat{s}_i < \overline{\overline{\tilde{v}}}_i \quad (8)$$

or

$$\overline{\overline{\tilde{v}}}_i < -m\sigma_{\tilde{v}_i} - K_\tau \hat{s}_i , \quad (9)$$

are met, where factors *m* and K_τ are dependent on the adopted confidence probability. Since the mean square deviation of the normalized innovation sequence is $\sigma_{\tilde{v}_i} = 1$, we can obtain the condition of absence of interval overlapping in more compact form combining (8) and (9):

$$\left|\overline{\overline{\tilde{v}}}_i\right| > m + K_\tau \hat{s}_i \quad (10)$$

Then the condition of overlapping the above intervals takes the form:

$$\left|\overline{\overline{\tilde{v}}}_i\right| \le m + K_\tau \hat{s}_i \, . \quad (11)$$

Thus algoritm 1, based on the check of testing conditions (10) and (11), allows the determination of technical condition of Kalman filter in real time. The fulfilment of condition (11) in this case means that priori and posteriori confidence intervals are overlapped, and consequently, Kalman filter is proper functioning. If condition (11) is not met, but (10) does, then the above confidence intervals are not covered, and hence, failure takes place.

5. THE CONSTRUCTION OF A TOLERANCE INTERVAL FOR $\tilde{v}_i$

If the interval of possible dispersion of data obtained from measurements is required to be estimated for further measurements using the same method under identical conditions, then tolerance intervals should be calculated. As is well- known [10], the interval which includes no less than the given part δ of the whole totality of data on a random value with the given probability γ is called the tolerance interval. The tolerance limit (γ ,δ) of tolerance interval $[l_1,l_2]$ means that 100% of δ for future observations will lie within this interval (see Fig.1) with 100% confidence probabilty γ. As a rule, they select values of δ and γ greater than 0.5. Since the checking parameter $\tilde{v}_i$ in this case is distributed by the normal law, then using sampling characteristics $\overline{\overline{\tilde{v}}}$ and $\hat{s}^2$, we can calculate the lower l_1 and upper l_2 tolerance boundaries for the normalized innovation sequence by formula [10]

$$l_1 = \overline{\overline{\tilde{v}}} - K_t \hat{s} \;;\quad l_2 = \overline{\overline{\tilde{v}}} + K_t \hat{s} \, , \quad (12)$$

where K_t is the tolerance factor.

To determine tolerance boundaries l_1 and l_2, values $\overline{\overline{\tilde{v}}}$ and $\hat{s}$ are to be calculated. Then K_t should be also obtained for which interval $[\,\overline{\overline{\tilde{v}}} - K_t\hat{s}; \overline{\overline{\tilde{v}}} + K_t\hat{s}\,]$ includes the given part δ of the totality of random value $\tilde{v}_i$ with confidence probability γ close to unity. Value K_t being the function of M, δ, and γ , can be approximately given by

$$K_t = z_\infty \left(1 + \frac{Z_\gamma}{\sqrt{2M}} + \frac{5Z_\gamma^{\;2} + 10}{12M} \right), \quad (13)$$

where Z_∞ and Z_γ are the abscissae of the normalized Laplace function $\Phi_0(Z)$ which can be determined from the conditions

$$\delta = 2\,\Phi_0(Z_\infty), \qquad \Phi_0(Z_\gamma) = \gamma - 0.5. \quad (14)$$

The tolerance interval is the interval for a random value, as distinct from the confidence interval which can be constructed in order to cover a non-random value. The determination of tolerance factor K_t is the most laborious calculation process. To obtain this factor, tables can be used from [10] or formula from [11].

6. FAILURE PREDICTION USING TOLERANCE INTERVALS

Let us consider the following approach to the failure prediction in Kalman filter which is based on the priori constructed confidence and tolerance intervals for the normalized innovation sequence using the overlapping condition. If the mentioned intervals are covered, then the true value $\tilde{v}_i$ can lie within any one of them. We can assume in this case that the estimation system is proper functioning. Otherwise, the system state is substantially distinct from nominal conditions, and failure should be supposed.Thus the problem of failure prediction can be reduced to the determination of a single point whose simultaneous presence in both intervals is necessary and sufficient condition for failure prediction. When failures are predicted in real conditions of Kalman filter operation, the proposed algorithm can be reduced to the following sequence of calculations.

Algorithm 2.

1. Choosing confidence probability, construct confidence priori interval [a,b] for the normalized innovation sequence using standard normal distribution quantiles.
2. Using (3) and (4) calculate Kalman's state estimate for the system and the value of normalized innovation sequence $\tilde{v}_i$ at step i.
3. Determine the values of sampling mean and variance according to formula (7).
4. Determine the boundaries of tolerance interval $[l_1 ,l_2]$ for $\tilde{v}_i$ using (12)-(13) and given γ and δ.
5. Test the overlapping conditions for intervals [a,b] and $[l_1,l_2]$. To do this, it is sufficient to check the fulfilment of the conditions of overlapping the intervals under study

$$\left|\overline{\overline{\tilde{v}}}_i\right| \le m + K_t \hat{s}_i \quad (15)$$

as well as the conditions of nonoverlapping

$$\left|\overline{\tilde{\nu}}_i\right| > m + K_t \hat{s}_i \,. \qquad (16)$$

If condition (15) is not met, failure in Kalman filter is possible. The decision is taken on the necessity and nature of controlling actions on the process of its operation before actual failure occurs. Otherwise, it should be assumed that Kalman filter will operate properly.

6. Repeat calculations beginning from step 2 for the next time instant.

Thus the use of the condition of overlapping the confidence priori and tolerance intervals allows to solve the problem of state prediction of Kalman filter under operation conditions.

Algorithm 2 can be also used for predicting testing of multidimensional Kalman filter. Prediction in this case should be realized individually for each channel and correlation between the channels should be ignored. Because of the fact that methods for building a tolerance region for vector random values are lacking at present, the use of the approach proposed without simplifications seems to be impossible for the case discussed.

In order to use the proposed algorithm for multidimensional Kalman filter, we introduce two hypotheses: H_0 means that Kalman filter operates properly and H_1 means that failure occurs in the estimation system. Next, the n-dimensional innovation sequence is divided into n one- dimensional sequences, and confidence $[a_j, b_j]$ and tolerance $[l_{j1}, l_{j2}]$, $j = \overline{1, n}$ intervals should be determined for each sequence. The problem of failure prediction in multidimensional Kalman filter falls into n one-dimensional problems which can be successively solved. The rule of making decision can be written as a result of checking multidimensional filter as follows:

$$H : \left|\overline{\tilde{\nu}}_{ij}\right| \le m + K_t \hat{s}_{ji}, \forall j \in [1, n],$$
$$H_1 : \exists j \in [1, n], \ \left|\overline{\tilde{\nu}}_{ji}\right| > m + K_t \hat{s}_{ji}, \qquad (17)$$

i.e. hypothesis H_o is adopted if failures have not been predicted in any one channel of multidimensional Kalman filter, and hypothesis H_1 is adopted if potential failure has been detected in any filter channel.

7. A NUMERICAL EXAMPLE AND DISCUSSION OF THE OBTAINED RESULTS

Let us consider the dynamic system

$$x_{i+1} = Ax_i + w_{i+1}, \quad y_{i+1} = Cx_i + \mathrm{v}_{i+1},$$

with parameters A= 0.95; C= 1; σ_w^2 = 0.36; σ_v^2= 1; and $\hat{x}_0 = x_0 = 0$. Assume that processing measurements are realized using Kalman filter (3) which allows the determination of state $\hat{x}_i$ and error variance P_i at the i-th step.

It is easy to show that using algorithm 1, we can detect failures and using algorithm 2 we can predict possible failures in Kalman filter. Choosing the width of "sliding window" M=20 as the state estimate is calculated, we can determine the values of the normalized innovation sequence $\tilde{\nu}_i$ and sampling parameters $\overline{\tilde{\nu}}_i$ and $\hat{s}_i$. Then the confidence interval for $\tilde{\nu}_i$ can be constructed. To do this, we can determine the value of multiplier γ = 0.9986 from the table of quantiles of standard normal distribution with m=2.4. In order to determine a posteriori confidence interval at M=20 and β=0.99, we can obtain K_τ=2.96 from the table of τ – distribution quantiles. The value of tolerance multiplier K_t=2.62 can be obtained at M= 20, δ= 0.9 and γ= 0.99 from the table given in [10].

The decision on the detection of failure in filter (3) was taken on the basis of inequality (10), and prediction of potential failure was realized using inequality (16).

i) Kalman fitler is functioning properly. The graph of statistical values of $\left|\overline{\tilde{\nu}}_i\right|$ is shown in Figure 2 and graph of innovation sequence $\tilde{\nu}_i$ is presented in Figure 3. As seen in Figure2, $\left|\overline{\tilde{\nu}}_i\right|$ is lower than the threshold, Hence H_0 hypothesis is judged to be true.

ii) A sensor failure at iteration 30, changing the mean value of the innovation sequence is considered as follows:

$$z_i = x_i + \sigma_V \times randn + 4, \ (k \ge 30).$$

The graph of statistical values of $\left|\overline{\tilde{\nu}}_i\right|$ is shown in Figure 4 when a constant shift occurs in the measurement channel. As seen from Figure 4, values $\left|\overline{\tilde{\nu}}_i\right|$ icrease rapidly beginning from the 31st step, and also (11) at the 139th step and (15) at the 58th step are not fulfilled. This confirms the fact that the algorithm 1 can detect failure in Kalman filter at the 139th step, and algorithm 2 fixes this failure at the 58th step. Thus before actual failure occurs at the 139th step, signal is formed at the 58th step indicating failure in the system.
Normalized innovation sequence $\tilde{\nu}_i$ in the case of constant shift in measurement channel is presented in Figure 5.

iii) A sensor failure at iteration 30, slowly increasing failure has been introduced into the system, namely: measurement values have been increased by adding parameter $r_i = r_{i-1} + 0.03$ at the initial value $r_o = 0.01$:

$$z_i = x_i + \sigma_V \times randn + r_i, \ (k \ge 30)$$

The graph of statistical values of $\left|\tilde{\bar{v}}_i\right|$ is shown in Figure 6 when a slowly increasing shift occurs in the measurement channel. As seen in Figure 6, until the sensor failure occurs $\left|\tilde{\bar{v}}_i\right|$ is lower than the failure detection and failure prediction thresholds, and when a failure occurs in the measurement channel, $\left|\tilde{\bar{v}}_i\right|$ grows rapidly, after 86 iterations it exceeds the "failure prediction threshold" and after 125 iterations "failure detection threshold". Hence H_1 hypothesis is judged to be true. Thus before actual failure occurs at the 155th step, signal is formed at the 116th step indicating potential failure in the system. This failure causes a change in the mean of the innovation sequence. Normalized innovation sequence $\tilde{v}_i$ in the case of slowly increasing shift in measurement channel is presented in Figure 7.

The proposed failure prediction approach allows the decision to be made on the necessity and nature of controlling actions on the operation process before actual failure takes place. Consequently, algorithm 2 can be considered as a peculiar kind of indicator of "undangerous disruption" (lowering efficiency, accuracy or stability, loss of a portion of stability margin) which detects potential failure and predicts the appearance of actual failure in the system.

8. CONCLUSION AND DISCUSSION

The use of the condition of overlapping of priori and posteriori confidence intervals for the innovation sequence of Kalman filter makes it possible to solve the problem of failure detection in the estimation system under the conditions of its practical operation. Based on the proposed conditions of overlappin confidence (a priori) and tolerance intervals testing Kalman filter state can be realized if potential failures are predicted.

The method proposed is well suited to operate detection of constant shifting and slowly varied failures, such as distortion in measuring instruments, gyroscope axes shifting in navigation systems, changes in object noise mathematical expectation or measurements etc. The advantage of this method is its simple geometric interpretation, which gives a clear estimate of the process under study. But it does not allow the detection of variance and (or) covariation of object noise or measurements, which can be considered as the disadvantage of the method.

The use of algorithms 1 and 2 in the systems of automatic control permits operative trouble-shooting, prevention of the use of false information in controlling systems.

In the problems of failure prediction in multidimensional Kalman filters inevitable errors take place when a multidimensional innovation sequence is replaced by one-dimensional sequences, because of neglecting correlation relations. The error values are distinct in different problems being solved, and their detailed study is a seperate problem.

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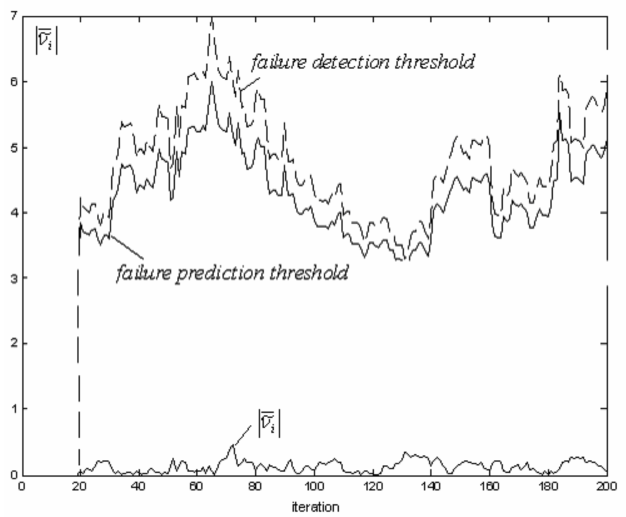

Fig. 2. Behaviour of $\left|\overline{\widetilde{\nu}_i}\right|$ when Kalman fitler is functioning properly

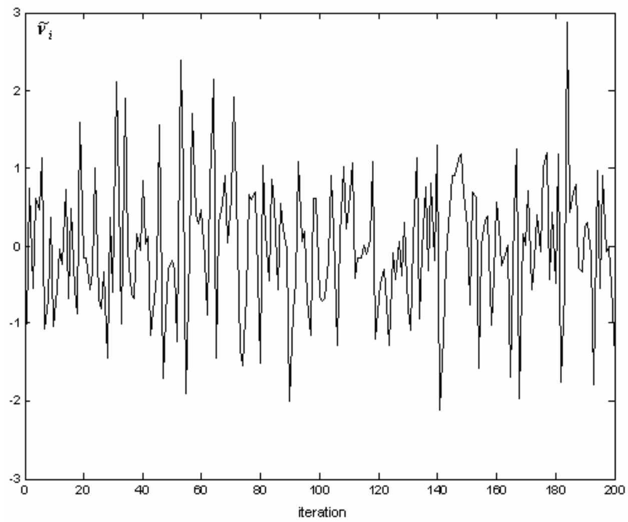

Fig. 3. Normalized innovation sequence $\widetilde{\nu}_i$ in the case when Kalman fitler is functioning properly

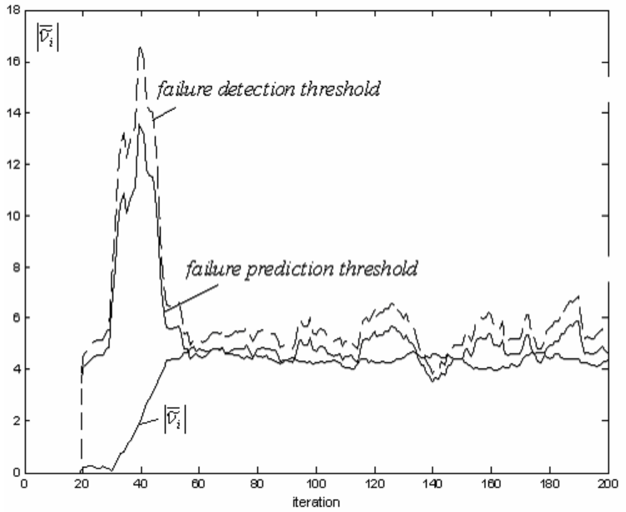

Fig.4. Behaviour of $\left|\overline{\widetilde{\nu}_i}\right|$ in the case of constant shifting in measurement channel

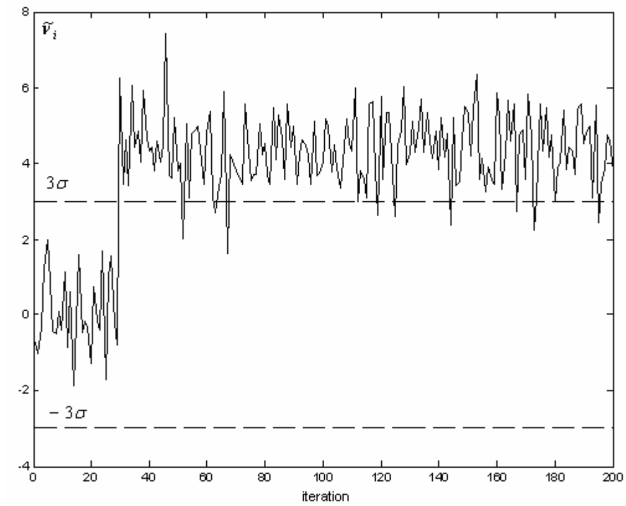

Fig.5. Normalized innovation sequence $\widetilde{\nu}_i$ in the case of constant shifting in measurement channel

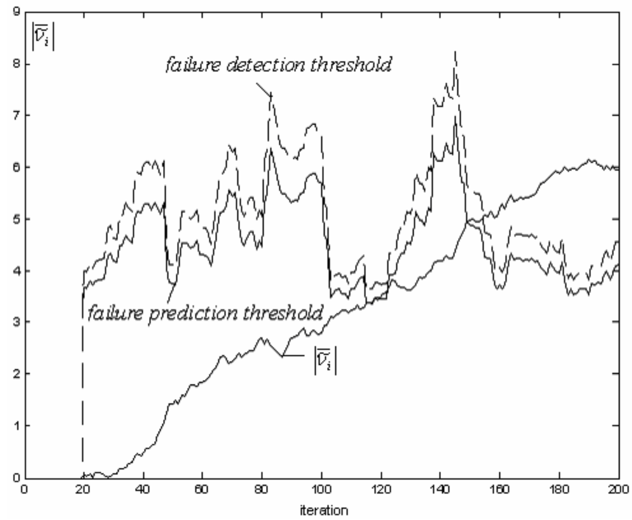

Fig. 6. Behaviour of $\left|\overline{\widetilde{\nu}_i}\right|$ in the case of slowly increasing shift in measurement channel

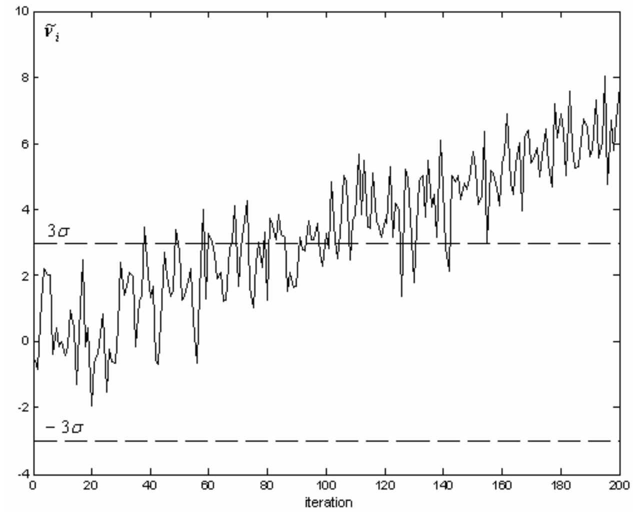

Fig.7. Normalized innovation sequence $\widetilde{\nu}_i$ in the case of slowly increasing shift in measurement channel