# Weighted mean and its uncertainty estimation

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**Abstract:** The weighted mean has been used to estimate the common mean of several populations with unknown and different variances. However, the traditional estimator of the variance of the weighted mean estimator underestimates the variance. Two new variance estimators are proposed with smaller biases and correspondingly formed intervals that have much better coverage probabilities for the mean. Results are extended to the general case with both Type A and Type B uncertainty components being presented.

## 1. INTRODUCTION

In statistics we consider a linear model for $k$ sets of random variables with a common mean such as

$$X_{ij} = \mu + \varepsilon_{ij}, \tag{1}$$

where $\mu$ is the common mean and the errors, $\varepsilon_{ij}$ 's for $i = 1,...,k$ and $j = 1,...,n_i$ are mutually independent and normally distributed with zero mean and variance $\sigma_i^2 (i = 1,...,k)$. The common mean, $\mu$ can be estimated by a weighted mean,

$$\bar{X}_w = \sum_{i=1}^{k} w_i \bar{X}_i, \tag{2}$$

where the weights $w_i$ 's satisfy $0 \le w_i \le 1$ and $\sum_{i=1}^{k} w_i = 1$, and $\bar{X}_i$ is the sample mean from the $i^{th}$ set of random variable, i.e.

$$\bar{X}_i = \sum_{j=1}^{n_i} \frac{X_{ij}}{n_i}. \tag{3}$$

From Graybill and Deal [1], $\bar{X}_w$ is the unbiased estimator of $\mu$ with a minimum variance among all the weighted means when the weights are

$$w_i = \frac{\frac{1}{\sigma_i^{'2}}}{\sum_{j=1}^{k} \frac{1}{\sigma_j^{'2}}}, \tag{4}$$

where $\sigma_i^{'2} = \sigma_i^2 / n_i$ provided that all the variances are known. In practice, however, $\sigma_i^2 (i = 1,...,k)$ are unknown. Thus, $w_i$ 's are usually estimated by

$$\hat{w}_i = \frac{\frac{1}{S_i^{'2}}}{\sum_{j=1}^{k} \frac{1}{S_j^{'2}}}, \tag{5}$$

where $S_i^{'2} = S_i^2 / n_i$ and $S_i^2$ is the sample variance estimated using $X_{ij} (j = 1,...,n_i)$. The corresponding weighted mean denoted by

$$\bar{X}_{\text{GD}} = \sum_{i=1}^{k} \hat{w}_i \bar{X}_i \tag{6}$$

is called the Graybill-Deal estimator of the common mean. In metrology, the Graybill-Deal estimator is often called the weighted mean. Weighted mean estimators such as the Graybill-Deal estimator have been used widely in practice to combine the results from several laboratories or based on several measurement methods. With the recent signing of the mutual recognition arrangement (MRA), national metrology institutes (NMIs) and regional metrology organizations (RMOs) around the world have committed to establishing the equivalence of measurement standards through key comparisons of national measurement standards. In key comparison studies, Graybill-Deal estimator or the weighted mean were often used to estimate the key comparison reference value (KCRV). It is well known that the variance of an estimator is as important as the estimator itself. In this paper, some new estimators of the variance of the Graybill-Deal estimator $\bar{X}_{\text{GD}}$ are proposed. In second section, some properties of the variance estimators of the Graybill-Deal estimators are discussed. In the third section, two new estimators of the variance of the Graybill-Deal estimator are proposed. Comparisons are made among various estimators of the variance of the Graybill-Deal estimator. In Section 4, results are extended to the general case with both Type A and Type B uncertainty components being presented. Conclusions are given in the last section. Throughout the paper, we assume that $\varepsilon_{ij}$ or equivalently $X_{ij}$ are independently normally distributed.

## 2. PROPERTIES OF THE VARIANCE OF $\bar{X}_w$ AND $\bar{X}_{\mathrm{GD}}$ AND THEIR ESTIMATORS

Under our normal assumption $\bar{X}_w$ is normally distributed with mean $\mu$. For $\bar{X}_w$ with weights $w_i$ defined in (4), the corresponding variance is

$$\mathrm{Var}[\bar{X}_w] = \frac{1}{\sum_{i=1}^{k} \frac{1}{\sigma_i^{'2}}}. \tag{7}$$

$\bar{X}_{\mathrm{GD}}$ defined in (6) is an unbiased estimator of $\mu$. Based on (7), many practitioners and metrologists e.g. [2], use the following statistic to estimate $\mathrm{Var}[\bar{X}_{\mathrm{GD}}]$

$$\widehat{\mathrm{Var}}[\bar{X}_{\mathrm{GD}}] = \frac{1}{\sum_{i=1}^{k} \frac{1}{S_i^{'2}}}. \tag{8}$$

We can show that

$$E[\widehat{\mathrm{Var}}[\bar{X}_{\mathrm{GD}}]] = E\left[\frac{1}{\sum_{i=1}^{k} \frac{1}{S_i^{'2}}}\right] \le Var[\bar{X}_w] \le Var[\bar{X}_{\mathrm{GD}}]. \tag{9}$$

The proof can be found in [3]. Thus, the traditional variance estimator $\widehat{\mathrm{Var}}[\bar{X}_{\mathrm{GD}}]$ in (8) underestimates both $\mathrm{Var}[\bar{X}_w]$ and $\mathrm{Var}[\bar{X}_{\mathrm{GD}}]$. In the next section, we propose two alternative estimators of $\mathrm{Var}[\bar{X}_{\mathrm{GD}}]$.

## 3. TWO ALTERNATIVE ESTIMATORS OF $\mathrm{Var}[\bar{X}_{\mathrm{GD}}]$

It is shown in [3] that

$$E\left[\frac{1}{S_i^2}\right] = \frac{(n_i-1)}{(n_i-3)}\frac{1}{\sigma_i^2}. \tag{10}$$

Thus, a modified estimator for $\mathrm{Var}[\bar{X}_w]$ and also for $\mathrm{Var}[\bar{X}_{\mathrm{GD}}]$ is given by

$$\widehat{\mathrm{Var}}_1[\bar{X}_{\mathrm{GD}}] = \frac{1}{\sum_{i=1}^{k} \frac{(n_i-3)}{(n_i-1)}\frac{1}{S_i^{'2}}} \tag{11}$$

It can be shown that

$$E\left[\frac{1}{\sum_{i=1}^{k} \frac{1}{S_i^{'2}}}\right] \le \frac{1}{\sum_{i=1}^{k} \frac{1}{\sigma_i^{'2}}} \le E[\widehat{\mathrm{Var}}_1(\bar{X}_{\mathrm{GD}})]. \tag{12}$$

The proof can be found in [3]. In particular, when $n_i = n$ for $i = 1,...,k$,

$$\widehat{\mathrm{Var}}_1(\bar{X}_{\mathrm{GD}}) = \left(\frac{n-1}{n-3}\right)\widehat{\mathrm{Var}}(\bar{X}_{\mathrm{GD}}).$$

Thus, $\widehat{\mathrm{Var}}_1[\bar{X}_{\mathrm{GD}}]$ and $\widehat{\mathrm{Var}}[\bar{X}_{\mathrm{GD}}]$ have a difference of a factor of $(n-1)/(n-3)$. For example, when $n = 20$, the ratio of these two estimators is 1.12. However, when $n$ is small, e.g., when $n$ = 5, the ratio is 2. That is, $\widehat{\mathrm{Var}}_1[\bar{X}_{\mathrm{GD}}]$ is twice as large as $\widehat{\mathrm{Var}}[\bar{X}_{\mathrm{GD}}]$. Thus, in estimating $\mathrm{Var}[\bar{X}_{\mathrm{GD}}]$, $\widehat{\mathrm{Var}}_1[\bar{X}_{\mathrm{GD}}]$ has a smaller bias than that of $\widehat{\mathrm{Var}}[\bar{X}_{\mathrm{GD}}]$.

Based on [4], using a second order approximation we propose another estimator of $\mathrm{Var}[\bar{X}_{GD}]$:

$$\widehat{\mathrm{Var}}_2[\bar{X}_{\mathrm{GD}}] = \frac{1}{\sum_{i=1}^{k} \frac{(n_i-3)}{(n_i-1)}\frac{1}{S_i^{'2}}}\left[1 + 2\sum_{i=1}^{k} \frac{\tilde{w}_i(1-\tilde{w}_i)}{n_i-1}\right] \tag{13}$$

where

$$\tilde{w}_i = \frac{\dfrac{(n_i-3)}{(n_i-1)}\dfrac{1}{S_i'^2}}{\sum_{j=1}^{k}\dfrac{(n_j-3)}{(n_j-1)}\dfrac{1}{S_j'^2}}. \qquad (14)$$

In particular, when $n_i = n$, $\tilde{w}_i = \hat{w}_i$

$$\widehat{\text{Var}}_2(\bar{X}_{\text{GD}}) \geq \widehat{\text{Var}}_1(\bar{X}_{\text{GD}}).$$

Simulations have also been done to compare the three estimators: $\widehat{\text{Var}}[\bar{X}_{\text{GD}}]$, $\widehat{\text{Var}}_1[\bar{X}_{\text{GD}}]$, and $\widehat{\text{Var}}_2[\bar{X}_{\text{GD}}]$. We used $n_i = 5$ and 15 and some variance structures of $\sigma_i^2$. First, the expectations of the variance estimators are compared with each other and with $\text{Var}[\bar{X}_{\text{GD}}]$. Based on limited simulations, the traditional variance estimator $\widehat{\text{Var}}[\bar{X}_{\text{GD}}]$ underestimates the variance of $\bar{X}_{\text{GD}}$ from 52-70 % when $n_i = 5$ and 18 to 24 % when $n_i = 15$. The estimator $\widehat{\text{Var}}_2[\bar{X}_{\text{GD}}]$ has smaller biases than those for $\widehat{\text{Var}}_1[\bar{X}_{\text{GD}}]$ while $\widehat{\text{Var}}_1[\bar{X}_{\text{GD}}]$ has a simpler form than that of $\widehat{\text{Var}}_2[\bar{X}_{\text{GD}}]$. Further we used simulations to compare the mean square errors of the four estimators of the variance of $\bar{X}_{\text{GD}}$. The mean square error of $\widehat{\text{Var}}[\bar{X}_{\text{GD}}]$ is defined as

$$\text{MSE} = E[\widehat{\text{Var}}(\bar{X}_{\text{GD}}) - \text{Var}(\bar{X}_{\text{GD}})]^2.$$

The mean square errors for other variance estimators are defined in the same way. Minimum MSE was used as another criterion to compare the estimators. The results show that for the same variance structures and the sample sizes when $k > 8$, $\widehat{\text{Var}}_2[\bar{X}_{\text{GD}}]$ has the smallest MSE's while the traditional variance estimator, $\widehat{\text{Var}}[\bar{X}_{\text{GD}}]$, has the largest MSE's for all the cases. When $k \leq 8$, $\widehat{\text{Var}}_1[\bar{X}_{\text{GD}}]$ has the smallest MSE's. Finally, we used simulation to compare the coverage probability of the intervals formed by the four estimators of the variance of the weighted mean estimator. Specifically, for $\widehat{\text{Var}}[\bar{X}_{\text{GD}}]$ and a coverage factor of 2, the $2\sigma$ interval is defined as

$$\bar{X}_{\text{GD}} \pm 2\sqrt{\widehat{\text{Var}}[\bar{X}_{\text{GD}}]}.$$

The coverage rate of this interval is the rate that the random interval covers the common mean $\mu$. The coverage rate is an estimate of the probability with which the mean $\mu$ is covered by the specified random interval. The other intervals are formed similarly. Using the coverage rate as the third criterion, with $n_i = 5$, it can be conclude that $\widehat{\text{Var}}_1[\bar{X}_{\text{GD}}]$ and $\widehat{\text{Var}}_2[\bar{X}_{\text{GD}}]$ perform better than the traditional variance estimator, $\widehat{\text{Var}}[\bar{X}_{\text{GD}}]$. In addition, we found that when the number $k$ increases, the coverage rate decreases for all the estimators. We also observed that when $n_i$'s are large enough such as 15, the coverage rates for all variance estimators are at least 0.92 for $2\sigma$ intervals and 0.987 for $3\sigma$ intervals. However, the intervals formed by $\widehat{\text{Var}}[\bar{X}_{\text{GD}}]$ always have the smallest coverage rates and the intervals formed by $\widehat{\text{Var}}_2[\bar{X}_{\text{GD}}]$ always have the largest coverage rates.

In summary, the simulations show that based on all three criterions $\widehat{\text{Var}}_1[\bar{X}_{\text{GD}}]$ and $\widehat{\text{Var}}_2[\bar{X}_{\text{GD}}]$ are better estimators than the traditional variance estimator. Between $\widehat{\text{Var}}_1[\bar{X}_{\text{GD}}]$ and $\widehat{\text{Var}}_2[\bar{X}_{\text{GD}}]$, the first one is computationally simpler than the second. On the other hand, the second one has smaller bias and larger coverage probability than that of the first one while the MSE's for these two estimators are comparable. In many cases, the biases for $\widehat{\text{Var}}_2[\bar{X}_{\text{GD}}]$ are positive, which is good in a conservative sense.

## 4. THE CASE WITH BOTH TYPE A AND TYPE B UNCERTAINTIES

Until now we consider the case that the uncertainties of $\{X_{ij}\}$ in (1) only consist of Type A uncertainties. The weighted mean and its uncertainty estimator were proposed in [2] for the case that both Type A and Type B uncertainty components presented. In this section, we will discuss the property of the uncertainty estimator in the presence of Type B uncertainties as well as the corresponding uncertainty estimators proposed in the previous sections. In parallel to the model in Section 1, we consider the situation in which the uncertainty of $\{X_{ij}\}$ or $\{\varepsilon_{ij}\}$ include both Type A and Type B uncertainties. As in [5] and [6], when

Type B uncertainties presented we assume that $X_{ij}$ can be expressed as

$$X_{ij} = X_{ij,A} + X_{i,B}, \tag{15}$$

where random variables, $X_{ij,A}$ and $X_{i,B}$ are statistically independent from each other, and their corresponding standard deviations or uncertainties are $\sigma_{i,A}$ and $\sigma_{i,B}$. The component $\sigma_{i,A}$ is the Type A uncertainty for the $i^{th}$ set of random variables and $\sigma_{i,B}$ is the Type B uncertainty for the same set of random variables. Thus,

$$\text{Var}[X_{ij}] = \sigma_{i,A}^2 + \sigma_{i,B}^2, \tag{16}$$

The sample means are thus

$$\overline{X}_i = \frac{\sum_{j=1}^{n_i} X_{ij}}{n_i} = \overline{X}_{i,A} + X_{i,B},$$

where $\overline{X}_{i,A} = \frac{\sum_{j=1}^{n_i} X_{ij,A}}{n_i}$. Thus,

$$\text{Var}[\overline{X}_i] = \frac{\sigma_{i,A}^2}{n_i} + \sigma_{i,B}^2 \equiv \sigma'^2_{i,A} + \sigma_{i,B}^2 \tag{17}$$

For the weighted mean, $\overline{X}_w$ defined in (2), the weights in (4) now become (using the same notation)

$$w_i = \frac{\frac{1}{\sigma'^2_{i,A} + \sigma_{i,B}^2}}{\sum_{j=1}^{k} \frac{1}{\sigma'^2_{j,A} + \sigma_{j,B}^2}}. \tag{18}$$

Similar to (7)

$$\text{Var}[\overline{X}_w] = \frac{1}{\sum_{i=1}^{k} \frac{1}{\sigma'^2_{i,A} + \sigma_{i,B}^2}}. \tag{19}$$

Although $\{X_{ij,A}\}$ are not observable, based on $\{X_{ij}, i=1,...,k;\ j=1,...,n_i\}$, $\sigma_{i,A}^2$ is estimated by the sample variance,

$$S_i^2 = \frac{\sum_{j=1}^{n_i} (X_{ij,A} - \overline{X}_{i,A})^2}{n_i - 1} = \frac{\sum_{j=1}^{n_i} (X_{ij} - \overline{X}_i)^2}{n_i - 1}.$$

Assume that $u_{i,B}^2$ is an unbiased estimator of $\sigma_{i,B}^2$, i.e. $E[u_{i,B}^2] = \sigma_{i,B}^2$. The uncertainty of the $\overline{X}_i$ is estimated by

$$\widehat{\text{Var}}[\overline{X}_i] = \frac{S_i^2}{n_i} + u_{i,B}^2 \equiv S'^2_i + u_{i,B}^2, \tag{20}$$

where $S'^2_i = S_i^2 / n_i$. Thus, the weights corresponding to (5) are given by (using the same notation)

$$\hat{w}_i = \frac{\frac{1}{S'^2_i + u_{i,B}^2}}{\sum_{j=1}^{k} \frac{1}{S'^2_j + u_{j,B}^2}}, \tag{21}$$

Then the weighted mean, $\overline{X}_{\text{GD}}$ corresponding to (6) is obtained. Now we consider the estimators of the uncertainty of $\overline{X}_{\text{GD}}$. Similar to (9), the following inequality holds,

$$\text{Var}[\overline{X}_{\text{GD}}] \geq \frac{1}{\sum_{i=1}^{k} \frac{1}{\sigma'^2_{i,A} + \sigma_{i,B}^2}}. \tag{22}$$

The proof is given in [3]. Similar to (10), an estimator of variance of $\overline{X}_{\text{GD}}$ is given by

$$\widehat{\text{Var}}[\overline{X}_{\text{GD}}] = \frac{1}{\sum_{i=1}^{k} \frac{1}{S'^2_i + u_{i,B}^2}}. \tag{23}$$

Similar to (9), we can show that

$$E\left[\frac{1}{\sum_{i=1}^{k} \frac{1}{S'^2_i + u_{i,B}^2}}\right] \leq \frac{1}{\sum_{i=1}^{k} \frac{1}{\sigma'^2_{i,A} + \sigma_{i,B}^2}}. \tag{24}$$

Thus, the estimator underestimates the variance of $\overline{X}_w$ and also the variance of $\overline{X}_{\text{GD}}$. The proof can be found in [3]. In parallel to (11) we have:

$$E[\hat{\mathrm{V}}\mathrm{ar}[\bar{X}_{\mathrm{GD}}]] \le \frac{1}{\sum_{i=1}^{k} \frac{1}{\sigma'^2_{i,A} + \sigma^2_{i,B}}} \le \mathrm{Var}[\bar{X}_{\mathrm{GD}}]. \quad (25)$$

From (10)

$$E\left[\frac{1}{\frac{n_i - 1}{n_i - 3} S'^2_i}\right] = \frac{1}{\sigma'^2_{i,A}}. \quad (26)$$

Since $E[u^2_{i,B}] = \sigma^2_{i,B}$ by the first order approximation

$$E\left[\frac{1}{u^2_{i,B}}\right] \approx \frac{1}{\sigma^2_{i,B}}. \quad (27)$$

Thus, from (26) and (27)

$$E\left[\frac{1}{\frac{(n_i - 1)}{(n_i - 3)} S'^2_i + u^2_{i,B}}\right] \approx \frac{1}{\sigma'^2_{i,A} + \sigma^2_{i,B}}. \quad (28)$$

From (28)

$$E\left[\frac{1}{\sum_{i=1}^{k} \frac{1}{\frac{(n_i - 1)}{(n_i - 3)} S'^2_i + u^2_{i,B}}}\right] \approx \frac{1}{\sum_{i=1}^{k} \frac{1}{(\sigma'^2_{i,A} + \sigma^2_{i,B})}}. \quad (29)$$

Based on that, similar to (11) an alternative estimator of the uncertainty of the weighted mean is given by

$$\hat{\mathrm{V}}\mathrm{ar}_1[\bar{X}_{\mathrm{GD}}] = \frac{1}{\sum_{i=1}^{k} \frac{1}{\frac{(n_i - 1)}{(n_i - 3)} S'^2_i + u^2_{i,B}}}. \quad (30)$$

It is clear that when $n_i > 3$,

$$\frac{1}{\sum_{i=1}^{k} \frac{1}{\frac{(n_i - 1)}{(n_i - 3)} S'^2_i + u^2_{i,B}}} > \frac{1}{\sum_{i=1}^{k} \frac{1}{S'^2_i + u^2_{i,B}}}. \quad (31)$$

From (31) $\hat{\mathrm{V}}\mathrm{ar}_1[\bar{X}_{\mathrm{GD}}]$ has a smaller bias than that for $\hat{\mathrm{V}}\mathrm{ar}[\bar{X}_{\mathrm{GD}}]$. Similar to (13), a second estimator of the uncertainty of the weighted mean is given by

$$\hat{\mathrm{V}}\mathrm{ar}_2[\bar{X}_{\mathrm{GD}}] = \frac{1}{\sum_{i=1}^{k} \frac{1}{\frac{(n_i - 1)}{(n_i - 3)} S'^2_i + u^2_{i,B}}} \left[1 + 2\sum_{i=1}^{k} \frac{\tilde{w}_i (1 - \tilde{w}_i)}{n_i - 1}\right] \quad (32)$$

where (using the same notation)

$$\tilde{w}_i = \frac{\frac{1}{\frac{(n_i - 1)}{(n_i - 3)} S'^2_i + u^2_{i,B}}}{\sum_{j=1}^{k} \frac{1}{\frac{(n_j - 1)}{(n_j - 3)} S'^2_j + u^2_{j,B}}}. \quad (33)$$

Since the factor $1 + 2\sum_{i=1}^{k} \frac{\tilde{w}_i (1 - \tilde{w}_i)}{n_i - 1}$ in (32) is larger than 1, $\hat{\mathrm{V}}\mathrm{ar}_2[\bar{X}_{\mathrm{GD}}] > \hat{\mathrm{V}}\mathrm{ar}_1[\bar{X}_{\mathrm{GD}}]$. It is clear that when both Type A and Type B uncertainties present, the proposed uncertainty estimators of the weighted mean has the benefit to reduce the bias. However, the improvement due to the proposed uncertainty estimators depends on the ratio between $S'_j$ and $u_{j,B}$ for all sets. When a Type B uncertainty component is equal to or larger than the corresponding Type A uncertainty component, the improvement due to $\hat{\mathrm{V}}\mathrm{ar}_1(\bar{X}_{\mathrm{GD}})$ is not as much as in the case when only Type A uncertainties exist.

## 5. CONCLUSIONS

The Graybill-Deal estimator or the weighted mean has been widely used to estimate the common mean from several sets of measurements, especially in key comparison studies recently. The widely used traditional variance estimator of the weighted mean estimator underestimates the variance and has large mean square errors and the intervals formed by this estimator have poor coverage probability for the mean, especially when the number of sets is large and the sample sizes are small. We have proposed two new estimators of the variance of the weighted mean when sample sizes are larger than 3. They have smaller biases, smaller MSE's in general, and the correspondingly formed intervals have much better coverage probabilities than those of the traditional variance estimator of the weighted mean. Overall, the second proposed estimator performs best. The results have been extended to the general case when both Type A and Type B uncertainty components are presented.

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