

PROPAGATION OF UNCERTAINTIES IN THE CALIBRATION CURVE FITTING OF SINGLE NORMAL HOT-WIRE ANEMOMETRY PROBES

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Abstract: The uncertainty propagation in the calibration of single normal hot-wire anemometry probes used in the constant temperature mode is discussed in this paper. The two most used forms of calibration curve (the fourth order polynomial and the King's law) are considered. The presented discussion is also useful for probes which are more complex than the single normal, such as the single yawed, the X, the V-shaped, the triple- and the four-wire probes. The uncertainties discussed here are important in mean flow velocity, and the discussion developed provides a notable opportunity to review some important aspects of the uncertainty propagation in any calibration involving curve fitting.

Keywords: calibration uncertainty, hot-wire anemometry, curve fitting.

1. INTRODUCTION

Hot-wire anemometry is a technique employed to measure instantaneous velocity or mean velocity and velocity fluctuation statistics in turbulent flow studies. It is based on convective heat transfer from an electrically heated wire or film element placed in a fluid flow. In the constant temperature mode of operation, a feedback differential amplifier is used together with a Wheatstone bridge circuit in order to maintain the hot-wire electrical resistance (and consequently its temperature) constant and generate an output voltage proportional to the power dissipated by the hot-wire [1].

Prior to its practical use, a hot-wire anemometry probe is subject to calibration at a nozzle calibration facility (or some other flow supply, like a wind-tunnel) with a variable reference (well-known) velocity and very low turbulence. The purpose of the present paper is to discuss the way the uncertainties in the data obtained from this calibration propagate into the result of a hot-wire anemometer measurement performed using one of the calibration equations considered here.

The difficulty of interpreting the effects of inclined velocities in relation to the hot-wire (and especially of

interpreting the effects of the three-dimensional velocity fluctuations associated with the turbulence) frequently takes up much of the attention of researchers involved with hot-wire anemometry uncertainty [2]. There are uncertainties due to the hot-wire probe dimensions (related to the spatial resolution of the measurements) [3], due to the frequency response of the anemometry system, and also due to the sampling rate, sampling time and analog/digital conversion resolution, when digital data acquisition is used. The uncertainties associated with electrical noise, probe positioning and climatic conditions (temperature, pressure and humidity), are also a cause of concern [4]. In order to focus the discussion, however, all these uncertainty sources will be disregarded here. Only the uncertainties in the calibration data and those arising from the calibration equation fitting process will be considered in the present paper.

The goal of such a simplified discussion is to reach a deep understanding of the uncertainties propagation in calibrations based on some different curve fitting processes, and then choose the best one. Besides being important for the mean velocity measurement, the present discussion is applicable to the calibration of many other types of sensors.

2. POLYNOMIAL CALIBRATION EQUATION

2.1. Using $E = f(V)$

The basic purpose of the hot-wire anemometry system calibration is to model the relationship between the input and output quantities. It may be seen as natural to treat the flow velocity V as the input signal and the voltage E at the differential amplifier output (in the constant temperature mode of operation) as the output. Using a fourth order polynomial, as shown in references [1,4], the calibration equation becomes

$$E = \sum_{i=0}^4 a_i V^i \quad (1)$$

To fit this equation to the calibration data a linear least squares regression is employed and the use of QR algorithm is recommended [5].

According to the ISO guide [6], the assessment of the uncertainty in the results obtained using the calibration curve is performed using

$$u_E^2 = \left(\frac{\partial E}{\partial V} u_{Vc} \right)^2 + \sum_{i=0}^4 \left(\frac{\partial E}{\partial a_i} u_{a_i} \right)^2 + \sum_{i=0}^3 \sum_{j=i+1}^4 \frac{\partial E}{\partial a_i} \frac{\partial E}{\partial a_j} \text{cov}(a_i, a_j) \quad (2)$$

which, for the considered polynomial, leads to

$$u_E^2 = \left[\left(\sum_{i=1}^4 i a_i V^{i-1} \right) u_{Vc} \right]^2 + \sum_{i=0}^4 (V^i u_{a_i})^2 + \sum_{i=0}^3 \sum_{j=i+1}^4 V^i V^j \text{cov}(a_i, a_j) \quad (3)$$

The uncertainty in the hot-wire voltage reading would be the positive square root of Equation (3). The variances ($u_{a_i}^2$) and covariances $\text{cov}(a_i, a_j)$ are calculated as explained in [5]. It is assumed that the random uncertainty is uniform in the calibration data, and that it is correctly represented by the standard deviation of the calibration points from the fitted curve. The contribution of systematic errors to the measurement result uncertainty is taken into account separately, in the u_{Vc} term. The value of uncertainty in the reference velocity (u_{Vc}) is taken from the nozzle facility calibration certificate.

This uncertainty estimate does not account for the dispersion of results due to random errors, like those caused by electrical noise or by a digitalization process, that are filtered by the fitting process. The uncertainty due to these random errors, that depends on the number of readings used to produce a measurement result, must be square summed to the result of Equation (3) (see section H.1 in [6]).

In the third column of Table 1 results of Equation (1) fitted to the calibration data given in the first two columns of the same table are shown. In the fourth column the standard uncertainties (evaluated using the Equation (3)) of the values in the third column are shown. In Figure 1 this regression curve is represented with the corresponding expanded uncertainty.

There are just ten calibration points in Table 1, and when they are used to estimate the five coefficients of a fourth order polynomial, only five degrees of freedom will remain associated with the standard deviation estimate based on the fitted curve (and therefore associated with the covariances of the fitted curve coefficients). The last two terms of Equation (3), which correspond to the uncertainty contribution of the curve fitting, are added to the contribution of the calibration

flow velocity, which is the first term. The fitting contribution has generally more than twice the magnitude of the first term. So the effective number of degrees of freedom associated with the standard uncertainty of the result (supposing that the contribution of the reference velocity was calculated with much more than nine degrees of freedom) is greater than nine and the expanded uncertainty for an approximately 95% level of confidence is obtained using the coverage factor of two. The expanded uncertainties depicted in Figure 1 are calculated using this coverage factor.

The calibration data set presented in Table 1 was adapted from that appearing in chapter 6 of reference [4]. The same reference suggests in section 31.1.2 that for a good dedicated calibrator, the standard uncertainty in the velocity would be given by

$$u_{Vc} = 0.01 V + 0.02 \text{ [m/s]} \quad (4)$$

where V is the velocity (in m/s) of the air-flow supplied by the calibrator. The values given by this expression will be used in the present work.

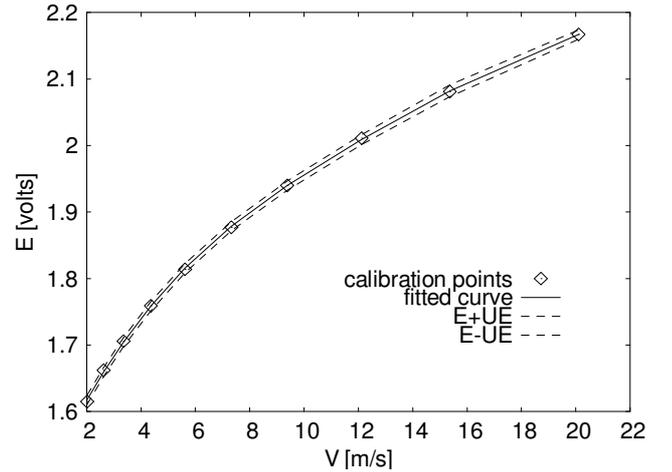


Fig. 1. Equation (1) fitted to calibration data with the corresponding uncertainty.

In the normal use of hot wire anemometry the output voltage E is known and one wishes to make a statement about a measured flow velocity V . In order to use a formula like Equation (1) in such a situation one must invert it. Since Equation (1) is a high order polynomial, it could not be analytically inverted. What is done is to use a numerical method to find a root of the function $f(V) - E$, in a range of V values that have physical significance.

Alternatively one could discard Equation (1) and fit a calibration curve of the kind $V = f(E)$ again using the least squares regression. Both options can be used and are discussed in the following items 2.2 and 2.3.

Table 1. Results for hot-wire calibration based on polynomial fitting.

Calibration data		$E = \sum_{i=0}^4 a_i V^i$		$\sum_{i=0}^4 a_i V^i - E = 0$		$V = \sum_{i=0}^4 a_i E^i$	
V [m/s]	E [volts]	E [volts]	u_E [volts]	V [m/s]	u_V [m/s]	V [m/s]	u_V [m/s]
2.019	1.615	1.618	0.003	1.979	0.049	2.011	0.042
2.622	1.662	1.659	0.003	2.665	0.051	2.642	0.047
3.358	1.706	1.705	0.003	3.383	0.058	3.348	0.054
4.360	1.759	1.758	0.003	4.372	0.070	4.360	0.064
5.621	1.814	1.816	0.003	5.584	0.083	5.613	0.076
7.324	1.877	1.878	0.003	7.280	0.101	7.330	0.094
9.379	1.940	1.940	0.004	9.381	0.129	9.378	0.114
12.121	2.011	2.009	0.004	12.210	0.158	12.129	0.142
15.364	2.081	2.082	0.004	15.307	0.202	15.355	0.174
20.101	2.167	2.167	0.004	20.116	0.304	20.103	0.221

2.2. Numerically inverting $E = f(V)$

Numerically inverting (or *solve* for the formerly independent variable V , or *find a root of*) an equation like Eq. (1) generally consists of a logically devised trial and error process by which successively better approximations to the input velocity V (that would satisfy the equation for a given value of the output voltage) are substituted into the equation. The trials are repeated until the difference between the E value resulting from the substitution and the given output voltage falls below a prescribed tolerance. Software packages like MATLAB offer well tuned implementations of the iterative algorithms that describe such processes (like those accessed in MATLAB using the intrinsic function “FSOLVE”) for solving nonlinear equations.

Now what is needed is a law of propagation of uncertainties (LPU) into the measured velocity obtained solving Equation (1) for V . Such an LPU is derived by interchanging the places of the symbols E and V in Equation (2). The partial derivatives of V needed for using the expression so obtained may be derived analytically using the rule of derivation of implicit functions [7,8] as, e.g.,

$$\frac{\partial V(E, a_0, \dots, a_4)}{\partial a_i} = - \frac{\frac{\partial E(V, a_0, \dots, a_4)}{\partial a_i}}{\frac{\partial E(V, a_0, \dots, a_4)}{\partial V}} \quad (5)$$

These derivatives could also be obtained numerically, but, in spite of the many automatic differentiation resources nowadays available, this does not seem to be a safer procedure than the analytical one for polynomial calibration curves. Derivatives obtained by the finite differences method can be coarsely wrong and care is needed in dealing with its round-off and truncation errors.

Using the rule of derivation of implicit functions one obtains

$$u_V^2 = u_{V_c}^2 + \left(\sum_{i=1}^4 i a_i V^{i-1} \right)^{-2} \left[u_E^2 + \sum_{i=0}^4 (-V^i u_{a_i})^2 + 2 \sum_{i=0}^3 \sum_{j=i+1}^4 V^i V^j \text{cov}(a_i a_j) \right] \quad (6)$$

The fifth and sixth columns of Table 1 presents, respectively, the results of numerically inverting Equation 1 (using the voltage E , measured in the calibration, as input data) and the velocity uncertainties evaluated using Equation (6). Figure 2 represents graphically these data in the same sense as the Figure 1 represents the data in the third and fourth columns of Table 1, but in the Figure 2 the fitted polynomial was used to obtain the abscissas correspondent to the adjusted curve (not the ordinates as done in Figure 1) and the uncertainties in Figure 2 correspond to displacements of these abscissa values.

In spite of these differences between the way in which Figures 1 and 2 were obtained, the curves shown are identical, which was expected. Both figures show the same calibration curve, as well as the limits of the region where 95% of the measurement results are expected to fall. They were obtained separately to verify the coherence between the calculation procedures used for presenting the calibration results (Figure 1) and for presenting measurement results obtained using the calibration curve (Figure 2).

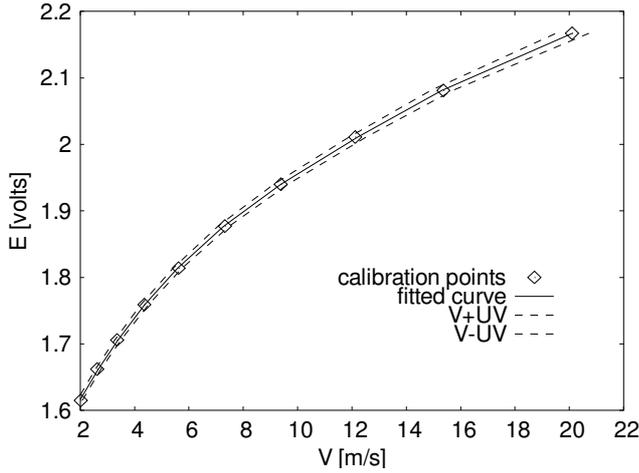


Fig. 2. The same data shown in Figure 1, obtained now using Equation (6) and a numerical inversion of Equation (1).

2.3. Using $V = f(E)$

If a calibration curve of the kind $V = f(E)$ is chosen, one basically needs to interchange the places occupied by the symbols E and V in the equations (1) and (3), but it is always important to analyze the meaning of each term in the LPU, in order to avoid the omission or the undue repetition of any important contribution to the estimated uncertainty.

The results of the fitted fourth order polynomial of the form $V = f(E)$, along with the uncertainty estimated by the analogue of Equation (3), are shown in the seventh and eighth columns of Table 1. In Figure 3 these data are plotted, again using expanded uncertainties.

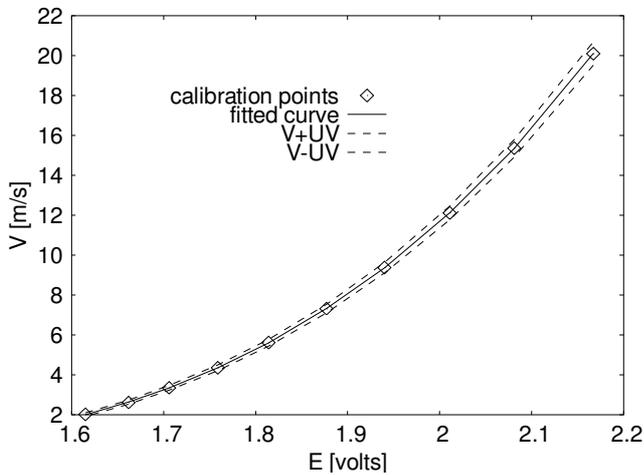


Fig. 3. The $V = f(E)$ curve fitting with its uncertainty.

Comparing the uncertainties shown in eighth and the sixth columns of Table 1 one can notice that the calibration data are better fitted to the $V = f(E)$ kind of polynomial, producing smaller uncertainties. But the fourth order polynomial used was flexible enough to satisfactorily represent the data no matter which variable is chosen as the independent one, as can be seen by the similarity between the curves in Figures 2 and 3.

3. THE KING'S LAW

Another calibration curve frequently used for hot-wire anemometry is the King's law

$$E^2 = A + B V^n \quad (7)$$

where A , B and n are parameters to be determined through the fitting process. Traditionally A and B were obtained fitting a straight line to $E^2 \times V^n$ data and many values of n are tested to choose the best one, starting with $n = 0.45$. If the best fit is obtained with a n too far from this, there is a great probability that coarse errors are contaminating the calibration data. Nowadays computer implementations of nonlinear least squares algorithms are widely available (for example, those accessed using the MATLAB intrinsic function "LSQCURVEFIT") and the fitting process is performed automatically. Equation (7) is easily inverted analytically, giving

$$V = \sqrt[n]{\frac{E^2 - A}{B}} \quad (8)$$

and there is no need to use numerical inversion algorithms.

3.1. Uncertainty propagation based on Taylor series

For the King's law a LPU analogue to Equation (2) is not readily applicable since there is no closed form analytical expression for the covariance matrix of the parameters fitted using a nonlinear least squares method. It is possible, however, to propagate the uncertainty in calibration data directly to the results of Equation (8) or (7), as suggested in [9] in a more comprehensive context. So, for Equation (8), taking the standard deviation of the calibration points from the fitted curve as their uncertainty, one obtains

$$u_v^2 = u_{Vc}^2 + \sum_{i=1}^m \left(\frac{\partial V}{\partial V_i} \sigma \right)^2 \quad (9)$$

The standard deviation σ is estimated as

$$\sigma = \sqrt{\frac{1}{m-3} \sum_{i=1}^m [V_i - V(E_i)]^2} \quad (10)$$

The partial derivatives are approximated by finite differences as

$$\frac{\partial V}{\partial V_i} \approx \frac{V(V_i + \varepsilon) - V(V_i)}{\varepsilon} \quad (11)$$

In these expressions m is the number of calibration points (E_i, V_i) . The number of degrees of freedom is equal to $m - 3$ because three parameters are estimated in the fitting process. $V(E_i)$ is the value calculated by Equation (8) when the value E_i substitutes the E appearing in it. $V(V_i)$ is the value calculated by Equation (8) using parameters fitted to the original calibration data. $V(V_i + \varepsilon)$ is the value calculated by Equation (8) using parameters fitted to the calibration data modified by the addition of a small increment ε to the i -th velocity used in calibration.

Table 2. Comparison of 4th order polynomial and King's law, using Taylor series (T.S.) and Monte Carlo (M.C.) approaches for assessing uncertainties of the latter calibration curve.

Calibration data		$V = \sum_{i=0}^4 a_i E^i$		$V = \sqrt[n]{\frac{E^2 - A}{B}}$ & T.S.		$V = \sqrt[n]{\frac{E^2 - A}{B}}$ & M.C.	
V [m/s]	E [volts]	V [m/s]	u_V [m/s]	V [m/s]	u_V [m/s]	V [m/s]	u_V [m/s]
2.019	1.615	2.011	0.042	2.005	0.040	2.005	0.040
2.622	1.662	2.642	0.047	2.642	0.047	2.642	0.047
3.358	1.706	3.348	0.054	3.351	0.054	3.351	0.054
4.360	1.759	4.360	0.064	4.363	0.064	4.363	0.064
5.621	1.814	5.613	0.076	5.615	0.076	5.615	0.076
7.324	1.877	7.330	0.094	7.329	0.093	7.329	0.093
9.379	1.940	9.378	0.114	9.376	0.114	9.376	0.114
12.121	2.011	12.129	0.142	12.128	0.141	12.128	0.141
15.364	2.081	15.355	0.174	15.356	0.174	15.356	0.174
20.101	2.167	20.103	0.221	20.104	0.221	20.104	0.221

In Table 2 the results obtained using Equations (8) and (9) are shown in the fifth and sixth columns and compared with the calibration data (first and second columns) and results of the use of a fourth order polynomial (third and fourth columns) discussed in section 2.3. In Figure 4 the results of Equations (8) and (9) are plotted, but again the uncertainties are expanded (multiplied by a coverage factor of 2).

The authors were concerned about these uncertainties estimates based on a first order Taylor series expansion of the variances and wished to check them. Higher order terms could be added to the Taylor series, but a Monte Carlo approach was considered a less cumbersome (albeit more computationally expensive) way of obtaining safer uncertainty estimates. Besides other advantages of the Monte Carlo method, it avoids the difficulty of choosing the ε to be used in the finite difference approximations to partial derivatives ($\varepsilon = 0.001$ was used to obtain the results presented in Table 2 and Figure 4).

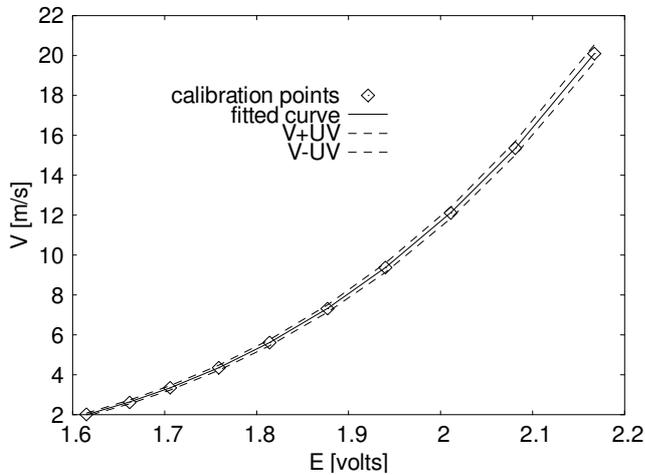


Fig. 4. The King's law curve fitting with uncertainties estimate based on Taylor series.

3.2. Monte Carlo simulation

In order to have better uncertainty estimates for the King's law results a Monte Carlo simulation of ten thousand calibrations was performed. Samples from a normal population of calibration velocities V_i were generated (for each E_i value) having the original V_i as mean and the result of Equation (9) as standard deviation. These samples were grouped as samples of simulated calibration data sets that lead to samples of fitted parameters sets (A , B and n). These samples of fitted parameters sets, in turn, generated samples of V values, (given an E value) when applied in Equation (8). The mean value of the latter samples (for each of the E values pertaining to the original calibration data set) is shown in the seventh column of Table 2 and its standard deviation is shown in the eighth one. Figure 5 is used to represent the results of this Monte Carlo simulation in the same way which Figure 4 was used for the results based on Taylor series expansion for variance.

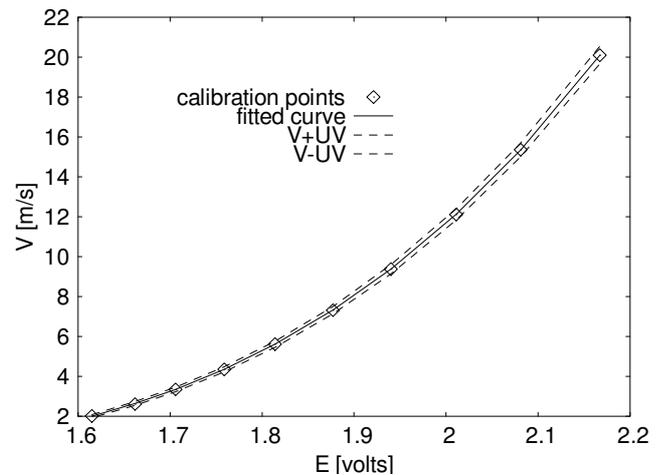


Fig. 5. The King's law curve fitting with uncertainties estimate based on M.C. simulation.

The results of Monte Carlo simulation confirmed those predicted by the first order Taylor series expansion of the

variance. The authors were afraid that the error propagation through a nonlinear fitting method could not be described by the linear LPU resulting from the first order Taylor series expansion of the variance, but this was not the case. The confirmed results show for the King's law uncertainties slightly smaller than those estimated for the fourth order polynomial, and this is in agreement with what is reported in section 4.5 of reference [1].

4. VELOCITY BIAS ERROR CANCELLATION IN TURBULENCE MEASUREMENTS

In face of the velocity standard uncertainties shown in Tables 1 and 2 one may wonder how low turbulence intensities could be measured with an instrument that, for a 95% level of confidence, would provide uncertainties between 2.2 % and 4.2 % of the measured velocity (the larger percentages corresponding to the lower velocities). The explanation for this relies on two important facts related to the propagation of calibration uncertainties into turbulence measurements. The first one is that turbulence levels are related to differences of velocities (instantaneous value minus the average – the so-called velocity fluctuation) measured using the same instrument and the same calibration curve. The second is that the errors contained in the calibration curve are systematic (what really matters, here, is that they are strongly correlated for velocities that are not too different from each other). This means that if the result of a velocity measurement is higher than the true velocity, the result of another measurement of a near velocity will be also higher than the true velocity by approximately the same amount.

When two quantities containing the same constant error are subtracted one from the other, the difference is free from that error¹. This is, essentially, what happens in turbulence intensity (and other statistic of velocity fluctuations) measurements when its value is small. It is possible to deal with this fact making careful estimates of the covariances of the two quantities being subtracted (section 4.2 of reference [9], and item F.1.2 of reference [6], provide some guidance for this estimation). This estimation, however, is far from being foolproof, and the error introduced by just disregarding the systematic uncertainties discussed here when performing turbulence intensity measurements (in low turbulence flows) is generally much smaller than the many other uncertainties mentioned at the introduction, but, for simplicity, not discussed in the body of the present paper.

5. CONCLUSION

The three calibration curves studied in present paper produced, as expected, small uncertainties when fitted by least squares to data representative of a single normal hot-wire anemometry probe calibration. The easiest to use of them is the fourth order polynomial which expresses velocities as a function of the output voltage: the fit can be performed using a simple linear least-squares algorithm, the uncertainties propagation in the fitting process is adequately

predicted with basis on a first order Taylor series expansion of the variance and there is no need for a nonlinear equation numerical solution to obtain the measurement results. The feeling that the flow velocity must be treated as an input of the anemometer and that the voltage as its output may be misleading in the calibration process.

The King's law produced uncertainties slightly smaller than those corresponding to the fourth order polynomial, as was to be expected from the physical basis on which it was developed. This advantage can be lost, however, if larger ranges of velocities are considered (see section 6 in [4]). Besides this, the numerical difficulties involved in the process of fitting the King's law to the calibration data and in the assessment of the uncertainty associated with it, although being of academic interest, may be good reasons to prefer a polynomial calibration curve.

In spite of the fact that the uncertainties evaluated in the present study are not very important in turbulence measurements, as explained in the section 4, they are relevant in measuring mean flow velocities and deserve attention.

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¹ This kind of error cancellation is discussed in section 5-2 of [9].