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A NOVEL METHOD OF ESTIMATING DYNAMIC MEASUREMENT ERRORS

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Abstract: A method for estimating an upper bound of the dynamic measurement error in the time domain is derived, starting from the transfer function of the system. Labeled *response uncertainty*, this dynamic error bound can be included in the conventional measurement uncertainty. A typical system for measuring force, pressure or acceleration is here evaluated using this measure. The linear dynamic error arises for two reasons: Varying amplification and delay with frequency. The latter is analogous to the well-known bandwidth limiting dispersion of signals in transmission systems. For wide spectrum signals/pulses the asymptotic tails of the spectra may generate the major part of the error. No widespread robust, general and systematic method of quantifying the measurement error caused by these effects exists, despite that they may generate signal distortion far beyond the common prediction.

Keywords: dynamic, uncertainty, error, distortion, delay

1. INTRODUCTION

Many measurement systems of today are statically calibrated with evaluation of the measurement uncertainty (MU) according to the GUM method [1]. When time-dependent signals are measured, the non-perfect dynamic response of the system generates additional errors. Only if the response time of the system is much shorter than the time scale of signal variation, dynamic effects will be negligible. In contrast to static calibration, dynamic calibration yields a *characterization* but not generally an accurate prediction of the maximum measurement error in the time domain. This characterization is often made in the frequency domain in terms of the complex-valued transfer function of the measurement system [2-4].

The linear dynamic signal error is a consequence of the total complex-valued transfer function of the system and the entire spectra of measured signals. Often only the maximum variation of the transfer function amplitude over the signal bandwidth is considered when estimating this error. Signal dispersion [5] related to the non-linear variation of the phase of the transfer function with frequency is then not considered. Although most correction methods aim at taking this effect fully into account [6-9], to our knowledge there has been no attempt to include it in estimates of the dynamic error. Also, the spectra of pulses are never strictly bandwidth limited and extend to infinity, even if the decay may be rapid. Proper evaluation of all these effects requires an unbounded summation/integration over frequency of a complex-valued function, weighted with a typical signal spectrum.

The present study proposes a method for finding a true upper bound of the linear dynamic error in the time domain, here denoted response uncertainty (RU). The transfer function is assumed known and can be obtained from a dynamic calibration, or by other means. All kinds of linear subsystems are allowed: Mechanical sensors for measuring pressure [2], force [3], or acceleration [4] as well as electrical and digital filters and amplifiers etc., are easily included in the analysis.

2. FORMULATION OF PROBLEM

The dynamic error is here defined as the maximum deviation of the observed signal $y(t)$ from a reference signal $y_0(t)$, translated in time by a possible delay τ. The only difference between these signals is that all dynamic effects are removed from y_0. Related to $y_M = \max(|y_0(t)|) \approx \max(|y(t)|)$, the error becomes a dimension-less function ε of the inferred time delay,

$$\varepsilon(\tau) \equiv \max_t \left(\left| \frac{y(t) - y_0(t-\tau)}{y_M} \right| \right). \tag{1}$$

Determination of the dynamic error *requires* knowledge of the time delay. From the transfer function it is not explicitly known. A generalized time delay between a given pair of signals can be defined using a correlation function technique. Alternatively, the delay can be obtained as the relative time translation minimizing the maximum signal difference. In simulations both methods can be used and compared. The latter method will be employed here since it directly addresses the dynamic error and gives the least error bound. In terms of the complex-valued frequency-dependent transfer function H,

$$
\begin{aligned}
x(t) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\omega)\exp(i\omega t)\,d\omega \\
y(t) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} H(i\omega)X(\omega)\exp(i\omega t)\,d\omega, \\
H(i\omega) &\equiv \left[H_0 + \delta\tilde{H}(i\omega,\tau)\right]\cdot\exp(-i\omega\tau)
\end{aligned}
\qquad (2)
$$

Capital letters will be used for the Fourier and Laplace transforms. The transfer function at zero frequency, $H_0 \equiv H(i\omega = 0) \in \boldsymbol{R}$, describes the linear response analyzed in static calibration, $y_0(t) = H_0 x(t)$. Here, $x(t)$ is the physical signal to be measured. For a linear phase system, $\delta\tilde{H} \in \boldsymbol{R}$. If also the amplitude $|H(i\omega)|$ is constant, $\delta\tilde{H} = 0$, there is no dynamic error.

The RU ε_D is an upper bound of the dynamic error $\varepsilon(\tau)$ in (1), considering all occurring signals $x(t)$ and minimizing over the delay τ,

$$
\varepsilon_D = \frac{1}{2\pi}\min_{\tau}\left[\max_{x(t),t}\left(\left|\int_{-\infty}^{+\infty}\frac{\delta\tilde{H}(i\omega,\tau)}{H_0}\frac{X(\omega)}{x_M}\exp(i\omega t)\,d\omega\right|\right)\right], \qquad (3)
$$

where $x_M = \max(|x(t)|)$. In the limit of a narrow spectrum around $\omega = \omega_B$, $X(\omega) \approx x_M \pi\delta(\omega^2 - \omega_B^2)$, a common error estimate results,

$$
\varepsilon_D(\omega_B) \approx \frac{\left||H(i\omega_B)| - |H_0|\right|}{|H_0|} \equiv \mu_D. \qquad (4)
$$

Even if not applicable, this is often used in practice, irrespective of the type of signal. The frequency ω_B is then chosen as the frequency component within the signal bandwidth that maximizes ε_D.

3. RESPONSE UNCERTAINTY

A general expression for the RU ε_D (section 3.1) is derived below. It depends on the product of bandwidth and duration of the signal (section 3.2). The asymptotic properties of the signal spectrum (section 3.3) and how it is modeled (section 3.4) are crucial aspects. In practice, the RU is preferably included in the MU (section 3.5).

3.1. General

An upper bound of the integral ε_D in (3) is found by moving the absolute value into the integrand,

$$
\begin{aligned}
\varepsilon_D(\omega_B) &= \gamma\min_{\tau}\left(\left\langle\left|\frac{\delta\tilde{H}(i\omega,\tau)}{H_0}\right|\right\rangle_B\right) \\
\langle f\rangle_B &= \frac{1}{\omega_B}\int_0^{+\infty} f(\omega)B(\omega)\,d\omega \\
B(\omega) &\equiv \frac{|X(\omega)|}{X_M} \le 1, \quad X_M \equiv \max_{\omega}(|X(\omega)|)
\end{aligned}
\qquad . \qquad (5)
$$

The bandwidth ω_B and the duration t_B of the signal measure the width of the signal spectrum and the persistence of the signal, respectively. Their product γ varies weakly with properties of the signal,

$$
\begin{aligned}
\gamma &\equiv \omega_B t_B \\
\omega_B &\equiv \int_0^{+\infty} B(\omega)\,d\omega \\
t_B &\equiv \frac{X_M}{\pi x_M} \approx \frac{Y_M}{\pi y_M}, \quad Y_M \equiv \max_{\omega}(|Y(\omega)|)
\end{aligned}
\qquad . \qquad (6)
$$

The frequency ω_B measures the signal bandwidth (in radians) similarly to commonly used bandwidths defined by the frequency at a given attenuation. The proposed bandwidth not only results in efficient and robust estimates of ε_D, it is also easily determined for *any* signal spectrum. If the spectrum consists of several more or less separated parts, the conventional bandwidth defined by any given attenuation is clearly not a useful concept.

Without *any* restriction on the signal, the dynamic error is truly unlimited. In each application, typical variations of the signal are usually approximately known. While an estimated rate of change is related to the distribution of the signal spectrum, the seldom known symmetry in time determines its phase. General statements can thus often only be made about the *relative magnitude* of the signal spectrum, as modeled by the *normalized* and *real*-valued *spectral distribution function* (SDF) B defined in (5). Different SDFs will correspond to different types of measurements. For each combination of application (B) and system (H), it is possible to evaluate the dynamic performance. The SDF here plays a similar role for generating dynamic errors as the probability distribution function (PDF) does for creating stochastic fluctuations in random processes. Both distributions describe the underlying variation that drives the system. How much of these that propagate to the measured quantity depend on the governing differential (algebraic) equations for the dynamic (stochastic) system. The algebraic equation should according to GUM [1] be linearized, while the differential equation is assumed to be linear. Many aspects are thus comparable and the abbreviation is chosen to reflect the equivalent status of the two.

For effective numerical calculation it is preferable to transform the improper integral of the weighted average in (5) to a bounded interval by changing variable, mapping $\omega = \{0, \omega_B, +\infty\}$ onto $\psi = \{1, 0.5, 0\}$,

$$
\langle f\rangle_B = \int_0^1 f\left(\omega_B\left(\frac{1}{\psi}-1\right)\right)B\left(\omega_B\left(\frac{1}{\psi}-1\right)\right)\psi^{-2}\,d\psi. \qquad (7)
$$

The signal $x_E(t)$ generating maximum error can be explicitly determined from the "worst" phase of the signal spectrum once the delay τ is known,

$$\frac{x_E(t)}{x_M} = t_B \int_0^{+\infty} B(\omega) \cos\left[\omega(t-t_0) - \arg\left(\delta\tilde{H}(i\omega,\tau)\right)\right] d\omega . \quad (8)$$

The time invariance is reflected by the arbitrary time t_0.

3.2. Bandwidth-duration product

From Fourier analysis methods [10] it is known that the value of the product γ of the bandwidth ω_B and the duration t_B varies little among different signals. Its value can be determined from the required harmonic limit (4) in conjunction with the integral (5) containing the transfer function variation,

$$\gamma_H = 1 . \quad (9)$$

This agrees with common estimates of bandwidth used in many engineering disciplines. Similarly, for both the Gaussian pulse $(\omega_1 = 0)$ and burst $(\omega_1 >> \alpha_1)$,

$$x(t) = \hat{x} \exp\left[-(\alpha_1 t)^2\right] \cos[\omega_1 (t - t_1)], \quad (10)$$

the bandwidth-duration product equals one. However, the expression (8) for the maximum error signal illustrates that γ can be larger than unity. If the system exhibits linear phase and $\delta\tilde{H}$ never change its sign the product γ_E is equal to unity, otherwise it is larger,

$$\gamma_E = \frac{\int_0^{+\infty} B(\omega) d\omega}{\max_t \left| \int_0^{+\infty} B(\omega) \cos\left[\omega t - \arg\left(\delta\tilde{H}(i\omega,\tau)\right)\right] d\omega \right|} \geq 1 . \quad (11)$$

The multi-component burst signal,

$$x(t) = \sum_n \hat{x}_n \exp\left(-\alpha_n^2 t^2\right) \cos(\omega_n t + \varphi_n) \equiv x_N(t), \quad (12)$$

also has a bandwidth-duration product γ_N slightly larger than one $\left(|\omega_j - \omega_k| >> \min(\alpha_j, \alpha_k), j \neq k\right)$,

$$\gamma_N = \frac{\sum_n \hat{x}_n}{\max_t \left(\left| \sum_n \hat{x}_n \exp\left(-\alpha_n^2 t^2\right) \cos(\omega_n t + \varphi_n) \right| \right)} \geq 1 . \quad (13)$$

The numerical value here depends on how well the harmonic components can be aligned. Reflecting the discussion above against the expected over-all accuracy, it nevertheless appears satisfactory to set $\gamma \approx 1$ for *all* signals.

3.3. Asymptotic behavior

Physical measurement systems have a finite bandwidth,

$$\delta\tilde{H}(i\omega) \to -H_0 \in \boldsymbol{R} < 0, \quad |\omega| \to +\infty . \quad (14)$$

The error transfer function $\delta\tilde{H}(i\omega)$ is neither integrable over $(-\infty, \infty)$, nor has a finite representation in time. A finite dynamic error thus *requires* a signal spectrum/SDF decaying to zero at infinity. An abrupt cut-off of the signal spectrum would however necessarily result in oscillations in the time domain. A finite asymptotic slope of the SDF is thus very important for correct modeling of real signals.

For studying convergence properties the RU is conveniently rewritten (D and $C >> \omega_B$ constants),

$$\int_0^{+\infty} \left| \frac{\delta\tilde{H}}{H_0} \right| B \, d\omega \approx D + \lim_{\Omega \to +\infty} \int_C^{\Omega} B \, d\omega . \quad (15)$$

If the SDF is of first order with a 6 dB/octave fall-off, $B \propto 1/\omega$,

$$\int_C^{\Omega} B \, d\omega \propto \ln\left(\frac{\Omega}{C}\right), \quad (16)$$

which diverges logarithmically. The divergence is quite slow implying that higher order SDFs B result in convergent integrals. The proposed calculation thus requires an asymptotic slope *strictly* exceeding first order (6 dB/octave). The mentioned bandwidth limitation of physical signals required to give finite dynamic errors in fact also makes them absolutely integrable. Then for n times differentiable signals,

$$\left| \frac{\partial^n x}{\partial t^n} \right| < +\infty \Leftrightarrow |X(\omega)| \leq \frac{|X(C)| C^{\alpha}}{\omega^{\alpha}}, \alpha > n+1, \quad \omega > C . \quad (17)$$

The minimum integer is $\alpha = 2$ for $n = 0$. The default slope for at least continuous signals is thus 12 dB/octave. The asymptotic properties of the SDF are primarily set by the regularity/ differentiability of the signals to be measured.

3.4. Determining the spectral distribution function

Finding a relevant SDF for calculating the RU is equivalent to determine the PDF for estimating the static MU. In the latter case, the selection is often based on the central limit theorem [10] while in the former there does not exist any equivalent theorem. Instead, it is the type of signal to be measured that determines the SDF: For pulses it will resemble the magnitude of low-pass filter transfer functions, for slowly varying harmonic signals it is bell-shaped and for periodic signals it may be a superposition of several such envelopes, etc.. Each application will have its own SDF and it is very important to state which has been used for evaluation. A basic assumption of the method is that the practitioner knows the application well enough to select a proper SDF.

Measured [output] signals are often accurate enough for estimating the unknown [input] SDF when the dynamic error is small. To show that a small error in the time signals corresponds to an equivalently small error in the frequency domain, Parseval's theorem can be utilized,

$$\int_{-\infty}^{+\infty} |y(t) - x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |Y(\omega) - X(\omega)|^2 d\omega . \quad (18)$$

A deviation between the input and output time signals of order ε_D leads to a similarly sized *mean* error between their frequency transforms. The unknown SDF for $X(\omega)$ can then be approximated with the SDF for $Y(\omega)$. The latter can be estimated from repeated measurements using the measurement system to be evaluated. Special attention must then be paid to how the system may change the asymptotic properties (section 3.3) of the SDF.

3.5 Total uncertainty

The RU ε_D can be included in a total uncertainty describing an upper bound of all linear measurement errors. The static sensitivity corresponds to the zero frequency limit of the transfer function while the variation of the transfer function with frequency sets an upper bound of the dynamic error. De-composing the transfer function into "static" and "dynamic" factors, its variation can be written as,

$$\frac{\delta H(i\omega)}{H} = \frac{\delta H_0}{H_0} + \frac{\delta \tilde{H}(i\omega)}{H_0}\exp(-i\omega\tau). \quad (19)$$

An upper bound of the total uncertainty is obtained by first completing the expression so that the last term equals the response uncertainty. Then averaging and using the triangle inequality,

$$\varepsilon_\Sigma = \varepsilon_S + \varepsilon_D. \quad (20)$$

The RU ε_D thus adds linearly with the MU ε_S to the total uncertainty ε_Σ. The quadratic sum rule based on stochastic uncorrelated error contributions, does not apply here. The uncertainty ε_S is to be determined according to GUM [1] for the chosen confidence level, exactly as in the corresponding static calibration. Since the RU is an absolute upper bound of the error, the total uncertainty inherits at least the confidence level of the static uncertainty. Formally, the RU can be accommodated in the uncertainty budget as an uncorrected deterministic type B contribution as described in section F.2.4.5 of [1].

4. TRANSDUCER EXAMPLE

Force and pressure transducers as well as accelerometers are usually [2-4] modeled as simple resonance systems,

$$H_T(s) = \frac{H_0}{(1 - s/p_0)(1 - s/p_0^*)}. \quad (21)$$

The resonance frequency $\omega_C = |p_0|$ sets the time scale. The dynamic performance of transducers can often be poor due to very low relative damping, $\zeta = -\mathrm{Re}(p_0)/|p_0|$. The transducer may be compared to similar dynamic systems having an equivalently defined damping. Electrical filters are often named according to their damping: Second order Bessel and Butterworth filters have the damping $\zeta = 0.87$ and $\zeta = 0.71$, respectively. In loudspeaker design [11] the group delay/phase characteristics of the system are very important to control in order to minimize waveform distortion. The damping can be tuned and is often chosen in the range $0.4 \le \zeta \le 1$. For instance, there exists a low damping "Boom-box design" which gives higher priority to constant amplification than linearity of phase. Correct levels of narrow spectra "tones" are then considered more important than accurate reproduction of pulses. This can be directly interpreted in the language introduced here: The SDF should be chosen as a localized function around the frequency of the tone rather than a low-pass filter for evaluating the pulse distortion. The expected duration of the tone sets the width of the SDF.

The generally very low damping of transducers (often $\zeta << 0.1$) is difficult to improve by physical means and may result in "ringing" effects [12-13] more severe than in any of the other mentioned examples. Some kind of correction filter H_C modifying the system transfer function, $H_T \to H_T \cdot H_C$, is therefore often required to reduce the measurement error. If no details about the transducer are known, conventional low-pass filters is a simple option. It is of course much more flexible, effective and accurate to apply a dedicated optimized [9] digital filter [14] on measured signals.

To analyze the transducer system with conventional low-pass correction filters, first select an SDF. Here assume that pulse measurements are of interest. Two different input signals will be simulated [15] to verify the RU: A quite irregular (only continuous) simple symmetric saw pulse and a very regular (infinitely differentiable) Gaussian pulse. The spectrum of the former is rather well described by the magnitude of a low-pass Bessel filter of second order, which is hence chosen as the SDF. The dynamic error of the latter signal is then expected to be much smaller than the RU due to its much higher regularity. The correction filter should be selected in accordance with the application, that is, the SDF. Measuring pulses an almost linear phase Bessel filter is a good choice. The order will more or less arbitrarily be chosen equal four, high enough to give sufficient high frequency damping but low enough for simple implementation and smooth roll-off to minimize filter distortion. With fixed order and type of the correction filter, the RU can be calculated for different filter cut-off frequencies ω_0 (see ε in Fig. 1.).

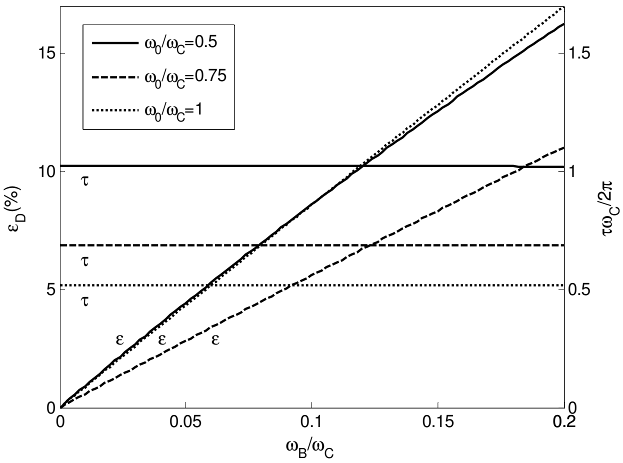

Fig. 1. The response uncertainty ε and time delay τ as function of bandwidth of a corrected transducer with relative damping $\zeta = 0.02$, for different values of the cut-off frequency ω_0 of the fourth order Bessel correction filter.

Lowering ω_0 will increase the delay of the measurement system. Often there is a trade-off between measurement accuracy and response time/delay, as can be seen in Fig. 1. Simulating measurements of the saw and Gaussian pulses for $\omega_0 = 0.75\omega_C$, the RU can be verified (Fig. 2.).

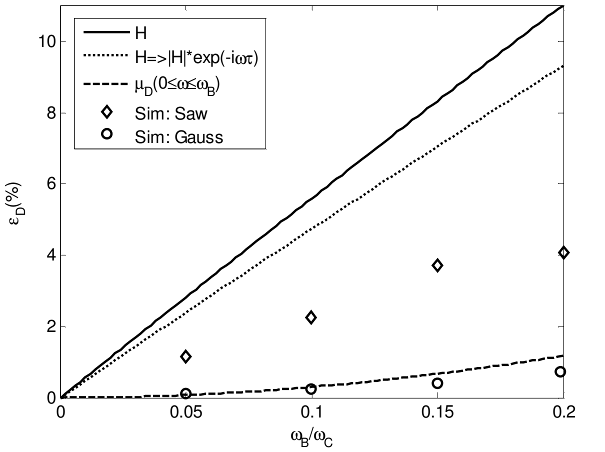

Fig. 2. The response uncertianty (full) as function of bandwidth of a corrected (fourth order Bessel, $\omega_0 = 0.75\omega_C$) transducer with relative damping $\zeta = 0.02$, compared to various approximations and simulations.

The difference between the common error estimate μ_D (broken) and the RU for the linearized phase system (dotted) is the inclusion of the asymptotic tails of the SDF. The RU (full) of the actual system with the non-linear phase, is here only slightly larger. Thus for ***this*** system, the dispersion is low compared to the dynamic error caused by the asymptotic tails of the SDF. The analyzed measurement system consisting of the weakly damped transducer and the Bessel filter, indeed has very little curvature in its phase. This is further illustrated by an almost constant time delay as function of bandwidth (see τ in Fig. 1). The common estimate (broken) here predicts a far too low error bound primarily due to the neglect of the important asymptotic tails of the pulse spectra/SDF. As seen from these simulations, only for such pulses as the Gaussian with very high asymptotic damping this estimate is accurate. For the saw pulse with $\omega_B = 0.1\omega_C$ the least error is, according to the simulations (RU), reached for a cross-over frequency of $\omega_0 = 0.77\omega_C$ $(0.75\omega_C)$ of the correction filter. Even if the RU is an upper bound of the simulated dynamic error, the deviation appears systematic.

5. CONCLUSION

A general method for estimating an upper bound of linear dynamic errors of measurement systems has been proposed and verified. Denoted *response uncertainty (RU),* it was shown to add linearly with the static uncertainty to give an estimate of the total measurement uncertainty. In the language of GUM, it can be included as a systematic uncorrected type B contribution. An example illustrated how subsystems of different characters easily are included to evaluate the total dynamic performance of a measurement system. The unprecedented *simultaneous*, *equivalent* and *general* analysis of the phase and amplitude characteristics of the whole system against the *entire* signal spectrum for estimating a dynamic error bound, is the hallmark of the RU.

In many respects the derived method parallels the well established stochastic GUM method for estimating the measurement uncertainty. It evaluates dynamic effects without making any modifications whatsoever to static error contributions. Due to this equivalence, simplicity and broad applicability it has a potential of becoming a widely spread method for estimating measurement errors in the time domain.

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