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# DETERMINATION OF THE POWER SPECTRAL DENSITY IN CAPACITIVE DIGITAL ACCELEROMETERS USING THEORY OF LIMIT CYCLES

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**Abstract:** Mechanical noise due to Brownian motion, electronic noise introduced by the interface circuit due to thermal noise sources in the electronic devices and quantisation noise due to the analog to digital conversion process are the main three noise sources presented in a sigma-delta modulator (ΣΔM) type system applied to a micromachined accelerometer: Based on theory of limit cycles in nonlinear closed loop systems, the mathematical model and formulae for power spectral density calculation of a micromachined accelerometer will be derived in the paper. The theoretical considerations will be verified with the simulation results in MatlabSIMULINK.

**Keywords:** sigma-delta modulator (ΣΔM), capacitive accelerometer, power spectral density.

## 1. INTRODUCTION

In a typical capacitive digital accelerometer, the proof mass is suspended above a substrate by compliant springs. Two nominally equal-sized sense capacitors are formed between the electrically conductive proof mass and stationary electrodes [1,2]. When the substrate undergoes acceleration, the proof mass displaces from the nominal position, causing an imbalance in the capacitive half-bridge, shown in Fig.1a. This imbalance can be measured using charge integration technique [1].

Force balancing of the proof mass is attained by enclosing the proof mass in a negative feedback loop. The feedback loop measures deviations of the proof mass from its nominal position and applies a force to keep the proof mass centered. The accelerometer output is taken as the force needed to null, or zero, the position, shown in Fig.1b.

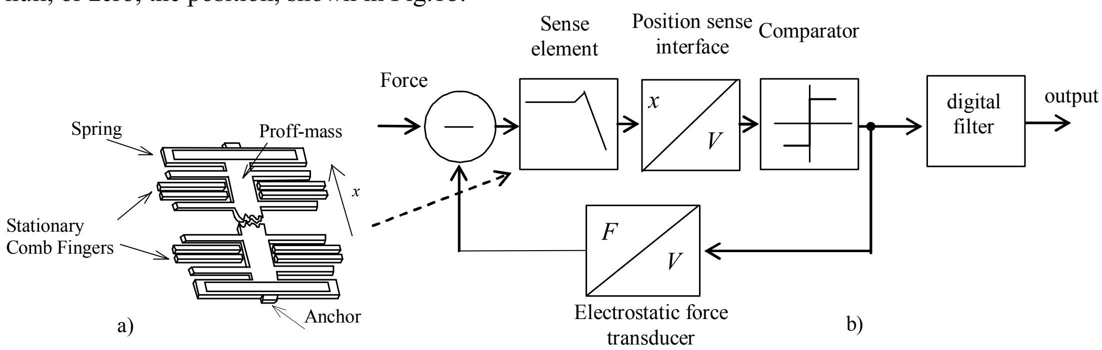



**Fig. 1a) Sense element, b) schematic of sigma-delta feedback loop**

The resolution and dynamic range of such a sensor are mainly determined by the signal to noise ratio of the system. A closed loop digital accelerometer suffers from three noise sources: 1) Brownian mechanical noise originating from the constant bombardment of air-molecules on the proof mass; 2) thermal noise from the electronic position measurement interface and 3) quantisation noise from the analogue to digital conversion process. The ΣΔM control system should be designed in such a way that the quantisation noise is much less significant than the other two noise sources since it is relatively easy to reduce the quantisation noise by increasing the sampling frequency of the modulator. It is well known that the sigma-delta modulator system shapes the quantisation and electronic noise in an advantageous way by attenuating it in the signal band.

Using theory of limit cycles, the mathematical model and formulae for determination of power spectral density of micromachined digital accelerometer is derived in the paper.

## 2. LINEARISED MODEL AND LIMIT CYCLES IN CAPACITIVE DIGITAL ACCELEROMETER

According to Fig. 1b, the effects of the comparator and sampler can be modeled by a complex sinusoidal dual-input describing function $N(\chi,\Phi)$ [3,4]. This approach is justified in this case since the sensing element effectively acts as a mechanical low-pass filter, which attenuates the harmonic components and quantisation noise $v_Q$ introduced by the comparator. As a consequence, the complexity of the mathematical model may be considerably reduced as shown in Fig. 2.

As in a true ΣΔM the unforced system is expected to exhibit a continuous oscillations. With the simplifications described above, it is now a relatively simple matter to predict these limit cycles conditions. This may be achieved by the solution of the system characteristic equation

$$1 + c_1 c_2 G(\mathrm{j}\omega) N(\chi, \Phi) = 0 \tag{1}$$

Here, $c_1 = T_f k_{fb}/(T_s V_{fb})$ and the describing function for the sampler and relay combination is given by [3,4]

$$N(\chi, \Phi) = \frac{4D}{\chi \pi} e^{-j\Phi}, \tag{2}$$

where $D$ denotes operating level of relay (in our case $D = V_{fb}$), $\Phi$ the lagging angle introduced by the sampling action and $\chi$ is the amplitude of the sinusoidal input signal to the comparator. By means of Eq. (1) Eq. (2) yields

$$1 + c_1 c_2 G(\mathrm{j}\omega) \frac{4D}{\chi \pi} e^{-j\Phi} = 0. \tag{3}$$

Since the mass of the sensing element must be maintained at the central position and for accurate coding the average error must approach zero, under this conditions the characteristic equation needs only by solved at specific frequencies $\omega = \omega_s/2n$, $n$ being the number of samples per half cycles of the sinusoidal input signal to the sampler. Limit cycle oscillations may only exist at these frequencies [5]. Graphical solution of Eq. (3) may be found in [6] however is not single valued. By solving Eq. (3) at the possible limit cycle modes an expression to predict the amplitude of input sinusoidal signal to comparator may be found

$$\chi = c_1 c_2 \left| G\left(\mathrm{j}\frac{\omega_s}{2n}\right) \right| \frac{4D}{\pi}. \tag{4}$$

As it will be mentioned later, a very efficiency tool to determine the frequency of input sinusoidal signal to comparator is MatlabSIMULINK.

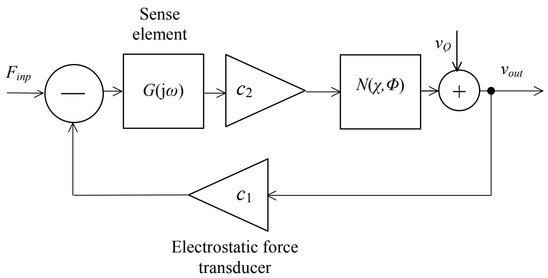



**Fig. 2 Simplified model of ΣΔM**

## 3. VALIDITY OF THE LINEARISED MODEL

Up to this point, the ΣΔM was considered to be unforced. In the unforced ΣΔM the amplitude of the sinusoidal signal $v_{out}$ representing limit cycles is $4V_{fb}/\pi$ and the mechanical low-pass filter attenuates other harmonic components and quantisation noise $v_Q$ introduced by the comparator as stated in the previous section. In order to predict the closed loop performance the effect of an input force $F_{inp}$ had to be investigated. In this case, the characteristic equation leads to

$$\left(\frac{F_{inp}}{c_1} - v_{out}\right) c_1 c_2 G\left(\mathrm{j}\frac{\omega_s}{2n}\right) N_{F_{inp}}\left(F_{inp}, \chi, \Phi\right) = v_{out}, \tag{5}$$

where $N_{F_{inp}}\left(F_{inp}, \chi, \Phi\right)$ is the triple-input describing function. By solving the Eq. (5), we obtain

$$N_{F_{inp}}\left(F_{inp}, \chi, \Phi\right) = \frac{v_{out}}{\left(\frac{F_{inp}}{c_1} - v_{out}\right)} \frac{1}{c_1 c_2 G\left(\mathrm{j}\frac{\omega_s}{2n}\right)}. \tag{6}$$

The voltage $v_{out}$ consists of two components, $F_{inp}/c_1$ due to the input force and $4V_{fb}/\pi$ due to the limit cycle. If, as it is normally found in practice, $F_{inp}/c_1 < 4V_{fb}/(10\pi)$ then the trial-input describing function $N_{F_{inp}}\left(F_{inp}, \chi, \Phi\right)$ approximates to the dual-input describing function $N(\chi, \Phi)$.

## 4. POWER SPECTRAL DENSITY AND EFFECTIVE NUMBER OF BITS OF ΣΔM

Linearised mathematical model of ΣΔM with a Brownian and an electronic noise sources is shown in Fig. 4. Here, the voltage signal $v_{out}$ is multiplied by $c_1$ to have the result in domain of measured force. Power spectral density of the Brownian noise is given by

$$N_B = 4 k_B R_M T, \tag{7}$$

where $k_B$ is the Boltzman constant and $T$ the temperature. Power spectral density $N_E$ of the electronic noise introduced by capacitive position measurement interface consists of two parts according to formula

$$N_E = N_{EA} + N_C, \tag{8}$$

where $N_{EA}$ is a power spectral density of a standard low-noise amplifier given in its datasheets and $N_C$ a power spectral density of an equivalent capacitance $C$. Power spectral density $N_Q$ of the quantization noise may be obtained from uniform probability of distribution and is given by

$$N_Q = \frac{2V_{fb}^2}{3 f_s}. \tag{9}$$

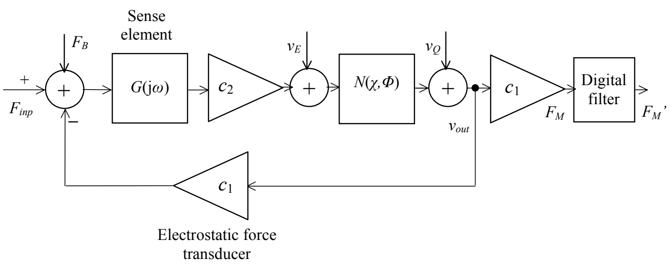



**Fig. 3 Simplified model of ΣΔM with Brownian and electronic noise sources**

It may be easily shown that the input measured force $F_M$ to digital filter is given by

$$F_M(\mathrm{j}\omega) = \frac{c_1 c_2 G(\mathrm{j}\omega) N(\chi, \Phi)}{1 + c_1 c_2 G(\mathrm{j}\omega) N(\chi, \Phi)} \left(F_{inp}(\mathrm{j}\omega) + F_B(\mathrm{j}\omega)\right) + \frac{c_1 N(\chi, \Phi) v_E(\mathrm{j}\omega)}{1 + c_1 c_2 G(\mathrm{j}\omega) N(\chi, \Phi)} + \frac{c_1 v_Q(\mathrm{j}\omega)}{1 + c_1 c_2 G(\mathrm{j}\omega) N(\chi, \Phi)} \tag{10}$$

in frequency domain. By means of Eq. (2) and (4) we receive that

$$\left|N(\chi,\Phi)\right| = \frac{1}{c_1 c_2 \left|G\left(\mathrm{j}\frac{\omega_s}{2n}\right)\right|}. \quad (11)$$

Because $\frac{|G(\mathrm{j}\omega)|}{\left|G\left(\mathrm{j}\frac{\omega_s}{2n}\right)\right|} \gg 1$ in the signal band, Eq. (9) yields

$$F_M(\mathrm{j}\omega) = F_{inp}(\mathrm{j}\omega) + F_B(\mathrm{j}\omega) + \frac{v_E(\mathrm{j}\omega)}{c_2 G(\mathrm{j}\omega)} + c_1 v_Q(j\omega)\frac{G\left(\mathrm{j}\frac{\omega_s}{2n}\right)}{G(\mathrm{j}\omega)}. \quad (12)$$

Eq. (12) shows that the measured force and Brownian noise remain unaffected while the quantisation and electronic noises are shaped differently by the ΣΔM. Considering all noise sources and measured force are mutually independent, the power spectral density of $F_M$ is given by

$$\mathrm{PSD} = \left|F_{inp}(\mathrm{j}\omega)\right|^2 + N_B + \frac{N_E}{\left(c_2\left|G(\mathrm{j}\omega)\right|\right)^2} + c_1^2 N_Q \left|\frac{G\left(\mathrm{j}\frac{\omega_s}{2n}\right)}{G(\mathrm{j}\omega)}\right|^2 \quad (\mathrm{N}^2/\mathrm{Hz}), (13)$$

where $N_B = F_B(\mathrm{j}\omega).F_B(-\mathrm{j}\omega)$, $N_E = v_E(\mathrm{j}\omega).v_E(-\mathrm{j}\omega)$ and $N_Q = v_Q(\mathrm{j}\omega).v_Q(-\mathrm{j}\omega)$. Typical bulk micromachined, capacitive accelerometers have sensing element with a critical damping ($(R_M C_M)^2 \approx 4mC_M$) [2], which leads to the following approximations:

$$\left|G(\mathrm{j}\omega)\right|^2 = 1, \quad \text{for } \omega < \frac{1}{\sqrt{mC_M}} \quad (14)$$

$$\left|G(\mathrm{j}\omega)\right|^2 = \frac{1}{\left(\omega^2 mC_M\right)^2}, \quad \text{for } \omega > \frac{1}{\sqrt{mC_M}}, \quad (15)$$

and

$$\left|G\left(\mathrm{j}\frac{\omega_s}{2n}\right)\right|^2 = \frac{16n^4}{\left(\omega_s^2 mC_M\right)^2}, \quad \text{because } \frac{\omega_s}{2n} \gg \frac{1}{\sqrt{mC_M}}. \quad (16)$$

where $G(j\omega)$ is the transfer function of the sensing element relating force to proof mass displacement as a second order mass-damper-spring ($m$ - $R_M$ - $1/C_M$) system. By means of Eq. (15), (15) and (16) the Eq. (13) yields

$$\mathrm{PSD}'_{\mathrm{dB}} = 10\log_{10}\left(\frac{\left|F_{inp}(\mathrm{j}\omega)\right|^2 + N_B + \frac{N_E}{c_2^2} + c_1^2 N_Q \frac{16n^4}{\left(\omega_s^2 mC_M\right)^2}}{F_{FS}^2}\right)$$

$$(\mathrm{dB/Hz}), \ \text{for } \omega < \frac{1}{\sqrt{mC_M}} \quad (17)$$

and

$$\mathrm{PSD}'_{\mathrm{dB}} = 10\log_{10}\left(\frac{\left|F_{inp}(\mathrm{j}\omega)\right|^2 + N_B + \frac{N_E}{c_2^2}\left(\omega^2 mC_M\right)^2 + c_1^2 N_Q 16n^4\left(\frac{\omega}{\omega_s}\right)^4}{F_{FS}^2}\right)$$

$$(\mathrm{dB/Hz}), \ \text{for } \omega > \frac{1}{\sqrt{mC_M}}, \quad (18)$$

where $F_{FS}=F_{fb}T_f/T_s$.

One of the most important parameters of ΣΔM is effective number of bits (ENOB) given by [1]

$$\mathrm{ENOB} = 1 + \log_2\left(\frac{F_{FS}}{\sqrt{3}F_{ef}}\right), \quad (19)$$

where $F_{ef}$ is the RMS of ΣΔM noise given by

$$F_{ef} = \frac{1}{\sqrt{2\pi}}\sqrt{\int_0^{\omega_m} N_B d\omega + \int_0^{(mC_M)^{-1/2}}\left(\frac{N_E}{c_2^2} + c_1^2 N_Q \frac{16n^4}{\left(\omega_s^2 mC_M\right)^2}\right)d\omega + \int_{(mC_M)^{-1/2}}^{\omega_m}\left(\frac{N_E}{c_2^2}\left(\omega^2 mC_M\right)^2 + c_1^2 N_Q 16n^4\left(\frac{\omega}{\omega_s}\right)^4\right)d\omega} \quad (\mathrm{N})$$

(20)

in the signal band up to $f_m$. It may be easily shown that by means of Eq. (9) the Eq. (19) leads to

$$F_{ef} = \sqrt{N_B f_m + \frac{4}{5}\left(\frac{N_E}{c_2^2} + c_1^2 N_Q \frac{16n^4}{\left(\omega_s^2 mC_M\right)^2}\right)\frac{1}{2\pi\sqrt{mC_M}} + \frac{N_E}{10\pi c_2^2}\left(mC_M\right)^2\omega_m^5 + \frac{c_1^2 V_{fb}^2 n^4}{15}\left(\frac{2f_m}{f_s}\right)^5} \quad (\mathrm{N}),$$

(21)

## 5. SIMULATION RESULTS

The MatlabSIMULINK simulation model of ΣΔM is depicted in Fig. 4. The parameters of the ΣΔM used during the simulation were as follows: $m$=1.7x10$^{-6}$ kg, $R_M$=5x10$^{-3}$ N(m/s)$^{-1}$, $C_M$ =0.1m/N, $V_{fb}$ = 5 V, $c_1$ = 0.5x10$^{-6}$ N/V, $c_2$ = 5x10$^4$ V/N, $T_f/T_s$=0.4 and $f_s$=1 MHz. At the beginning, the dual-input describing function was verified. The value of input force, electronic and Brownian noises were set to zero during the simulation. the amplitude of sinusoidal output force of mechanical filter was 1.85x10$^{-9}$ N and frequency $f_s/(2n)$=15 873 Hz ($n$=32). The theoretically predicted value ($\chi/c_2$) found by means of Eq. (4) was 1.89 x10$^{-9}$ N, which agreed well with the value predicted by simulation. In order to verify the Eq. (17) and (18) a few simulation experiments were made. In the first series of experiments, only the amplitude of input force was changed while Brownian and electronic noise sources remained equal to zero. Fig. 5 shows the simulated and theoretically predicted power spectral densities of output bitstreams related to full-scale feedback force. The input signal was assumed to be sinusoidal with a frequency 1.59 kHz and an amplitude of 0.22 μN in Fig. 5a and 0.02 μN in Fig. 5b. The power spectral density noise floor calculated by means of Eq. (17) was in both cases equal to – 127 dB/Hz. As it is obvious from Fig. 5, the simulated power spectral density approximates this value well. In the second series of experiments, an

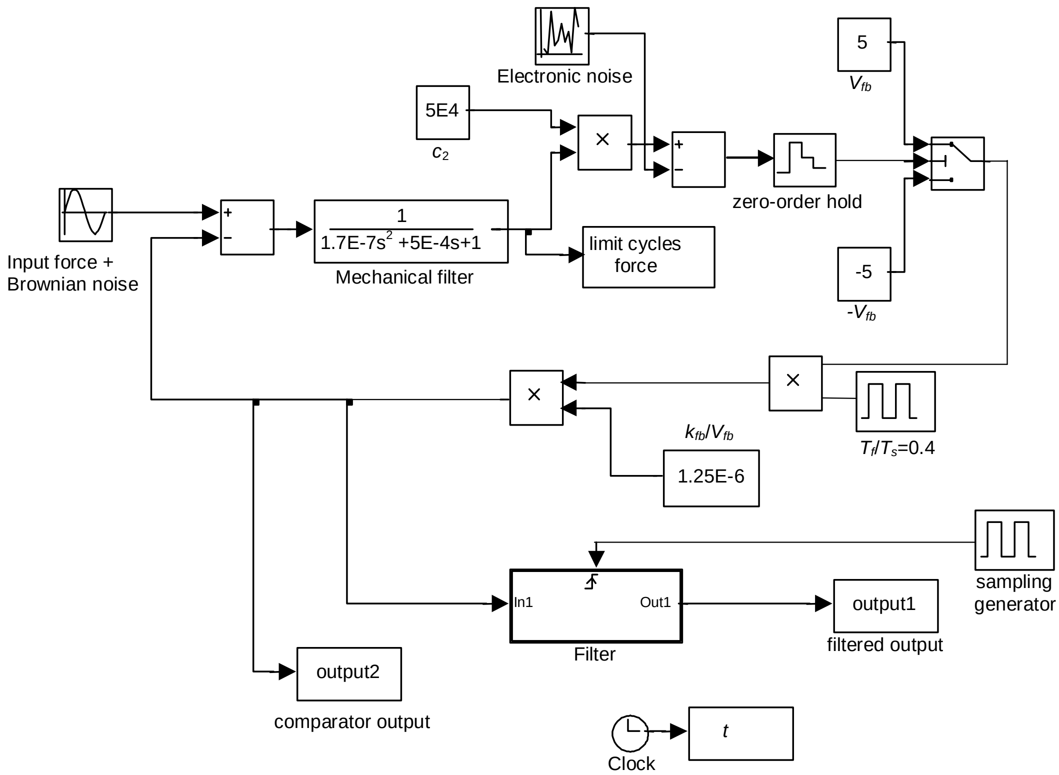
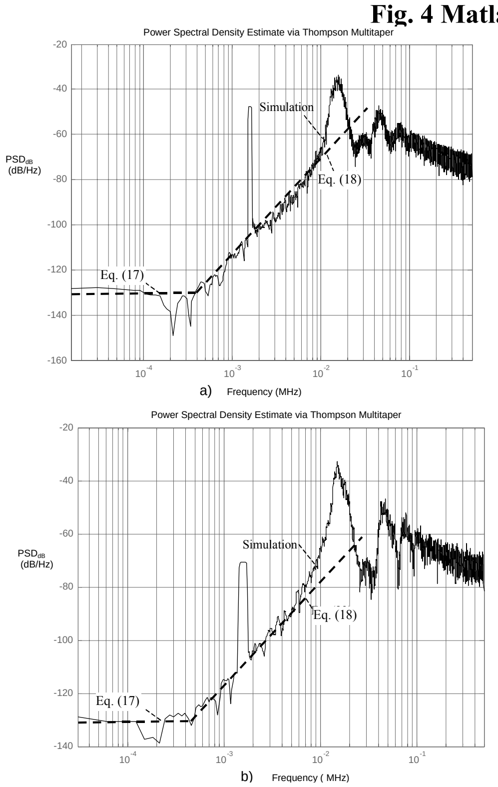



**Fig. 4 MatlabSIMULINK model of ΣΔM**



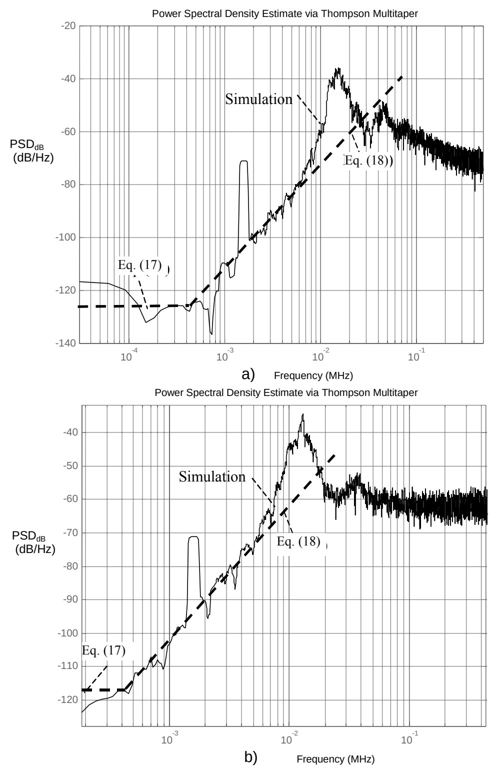




**Fig. 5 Simulated and theoretically predicted power spectral densities. The input signal was assumed to be sinusoidal with a frequency 1.59 kHz and an amplitude of a) 0.22 μN b) 0.02 μN**

**Fig. 6 Simulated and theoretically predicted power spectral densities. The input signal was assumed to be sinusoidal with a frequency 1.59 kHz and an amplitude of 0.02 μN and power spectral density of electronic noise equal to a) 5.3x10$^{-16}$ V$^2$/Hz b) 2.6x10$^{-14}$ V$^2$/Hz**

electronic noise was added to SIMULINK model. The amplitude of input sinusoidal signal was 0.02 μN and frequency 1.59 kHz. The simulated and theoretically predicted power spectral densities of output bitstreams related to full-scale feedback force are shown in Fig. 6. The power spectral density of electronic noise $N_E$ was assumed to be $5.3 \times 10^{-16}$ $V^2$/Hz in Fig. 6a and $2.6 \times 10^{-14}$ $V^2$/Hz in Fig. 6b.

## 6. CONCLUSIONS

The formulae derived in this work allow a structured approach to the design of a ΣΔM control system for micromachined sensors. The Brownian noise remains unaffected while the electronic and quantisation noise are shaped differently by the ΣΔM. Also, this work will allow to predict which noise source is dominant for different operating conditions. Consequently, the most important design parameter for the ΣΔM, the sampling frequency, can be chosen so that the quantisation noise level lies well below the other noise sources. For high oversampling ratios, Brownian noise will dominate as it is not shaped by the modulator. If the achieved ENOB is not sufficient for the sensor specifications, packaging at reduced pressure has to be considered as this lowers the Brownian noise level.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