XVIII IMEKO WORLD CONGRESS
Metrology for a Sustainable Development
September, 17 – 22, 2006, Rio de Janeiro, Brazil

GENERALIZED NONSUBTRACTIVE DITHERING IN A/D CONVERSION – THEORY AND SIMULATION RESULTS

Tomasz Adamski

Institute of Electronic Systems, Warsaw Institute of Technology, Warsaw, Poland, T.Adamski@ise.pw.edu.pl

Abstract: The paper deals with mathematical fundamentals of the generalized nonsubtractive dithering method (which can be used in A/D converters and sampling systems with repetitive sampling). The new concept is based on inverting a "dithering characteristic" F defined in the paper for arbitrary (but fixed) quantization functions. Errors of the method are assessed and analyzed in detail.

Keywords: dithering, A/D conversion, quantizers, measurement accuracy, stochastic processes

1.INTRODUCTION

It is well known (see [1-4]) that dither (i.e. noise purposely added to the converted input voltage) can improve accuracy of A/D converters. In general dithering methods are divided into two categories: a) subractive dither (when we subtract dither realization after quantization) and b) non-subtractive dither (when we do not subtract dither realization after quantization). The paper deals only with the non-subtractive dither.

In the paper we try to assess errors and accuracy of the classical dithering method and we propose a generalized dithering method. The essence of the new approach consist in averaging and inverting a "dithering characteristic" (defined in the sequel).

Every constant inside intervals, real function of a real variable $QN : R \to R$ is called a quantization function. From electronic engineering point of view the quantization function is a static input/output characteristic of an electronic circuit (with one or more comparators) called a quantizer. It is worth to underline that in the sequel $QN : R \to R$ does not have to be a uniform quantization function. Let D be a real random variable describing dither. Assume D is defined on a probabilistic space $(\Omega, \mathcal{F}, P)$ and $QN(a+D) \in L^1(\Omega, \mathcal{F}, P)$ for every $a \in R$. A function $F : R \to R$ given for every $a \in R$ with the formula (1) is called a "dithering characteristic" in the paper. P_D is a probability distribution of the random variable D.
An exemplary dithering characteristics is shown in the Fig. 2 (the dither distribution is Gaussian with the mean value 0

$$F(a) \overset{df}{=} E(QN(a+D)) = \int_R QN(a+x) P_D(dx) \qquad (1)$$

and a standard deviation $\sigma = 0.1\Delta x$, quantization function is uniform with $\Delta x = 1\,\mathrm{V}$). In the section 2 of the paper we assess properties of this function. In particular we prove that under natural assumptions the function F exists, is continuous and is strictly monotonic increasing (see Fig.1). The crucial property for our aims is possessing an inverse of the dithering characteristic. Some particular examples of dithering characteristics are given in the section 2.

A simplified block diagram of the circuit for measurement (i.e. A/D conversion) with generalized dithering method is shown in the Fig. 1. The dither D is added to the measured input voltage a and then the sum $a + D$ is quantized. The procedure is repeated N_0 times. Then quantized samples are averaged. The last step is correction of the average with the dithering characteristic F. Finally we obtain a random variable $A(N_0)$ which is an estimator of the value a.

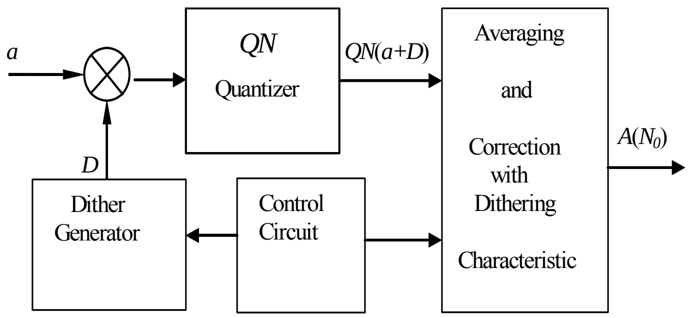

Fig.1. Addition of the dither D in the A/D conversion system,

The section 3 deals with generalized dithering methods. It is worth to underline that the method works for a large class of quantization functions $QN : R \to R$ and dither distributions P_D. The section 4 is devoted to errors and accuracy of the dithering methods. We finish the paper with some conclusions.

2. FUNDAMENTAL PROPERTIES OF DITHERING CHARACTERISTICS

In the section we give some basic properties of the dithering characteristic defined in the section 1. We would like to know if the dithering characteristic is monotonic increasing, strictly monotonic increasing or continuous.

Every constant inside intervals, monotonic increasing real function of a real variable $QN : R \to R$ is called a quantization function. More precisely, we assume in the definition that we have a sequence $(I_i)_{i \in Z}$ of intervals $I_i \subset R$ that for every $i \in Z$ and $x \in I_i$ we have $QN(x) = const.$ and $\bigcup_{i \in Z} I_i = R$ and there is a real number $\varepsilon > 0$ that for every $i \in Z$ we have $l_1(I_i) > \varepsilon$, where l_1 is a Lebesgue measure on *R*. Additionally we assume that for every $i, j \in Z$, $i \neq j$ we have $I_i \cap I_j = \emptyset$ and there exist two points $x_1, x_2 \in R$ that $QN(x_1) \neq QN(x_2)$.

Theorem 2.1. Assume $QN : R \to R$ is a quantization function. If a random variable D has the mean value i.e. $D \in L^1(\Omega,$ [illegible]$)$ then

1) for every $a \in R$ there is the mean value $E(QN(a+D))$ (then the dithering characteristic is well defined)

2) a dithering characteristic is always a monotonic increasing function

Proof. see [3] ■

Properties 1) and 2) are independent from the distribution of the random variable D describing dither, the distribution can be for example discrete, continuous or have an arbitrary type of the distribution. In particular $QN : R \to R$ can be a uniform quantization function.

Theorem 2.2. Let $QN : R \to R$ be a quantization function. If a random variable D has the mean value i.e. $D \in L^1(\Omega,$ [illegible]$)$ and has a density function then

1) for every $a \in R$ there is a mean value $E(QN(a+D)$

2) the dithering function

$$R \ni a \to F(a) = E(QN(a+D) \in R$$

is continuous and monotonic increasing.

Proof. see [3] ■

Theorem 2.3. Assume $QN : R \to R$ is a uniform quantization function. If a random variable D has the mean value i.e. $D \in L^1(\Omega,$ [illegible]$)$ and has the density function $f \in L^1(R,$ [illegible]$)$ such that for every $x \in [-\frac{1}{2}\Delta x, \frac{1}{2}\Delta x]$ (Δx is a parameter of the quantization function) we have $f(x) > 0$ then

1) for every $a \in R$ there is a mean value $E(QN(a+D)$

2) dithering characteristic

$$R \ni a \to F(a) = E(QN(a+D) \in R$$

is continuous and strictly monotonic increasing

Proof. see [3] ■

For the given quantization function $QN : R \to R$ and probability distribution of the random variable D we can compute values of the dithering characteristic $F : R \to R$ with arbitrary accuracy.

For simplicity reason we assume in the sequel that $QN : R \to R$ is a uniform quantization function. defined with the formula $QN(x) = \Delta x \left\lfloor \frac{1}{\Delta x} x + 1/2 \right\rfloor$ where $\Delta x > 0$ It is easy to note that $QN(x) = x - ([x + \Delta x/2]_{\Delta x} - \Delta x/2)$ then the quantization function is a sum of a linear function $id : R \to R$ and a periodic function $R \ni R \to -([x + \Delta x/2]_{\Delta x} - \Delta x/2) \in R$ with a period Δx. Therefore the value of the dithering characteristic $F(a)$ is equal to

$$\begin{aligned} F(a) &= E(QN(a+D) = \\ &= E(a+D) - E([a+D+\Delta x/2]_{\Delta x} + \Delta x/2 = \\ &= a + E(D) - E([a+D+\Delta x/2]_{\Delta x} + \Delta x/2 \end{aligned}$$

then the function F is also composed from the linear part and the periodic part with a period Δx. Then for computation of the dithering characteristic *F*, it is sufficient to know only the following function

$$[-\Delta x/2, \Delta x/2] \ni a \to E([a+D+\Delta x/2]_{\Delta x})$$

In practice linearity (or nonlinearity) of the dithering characteristic is important. Nonlinearity of the dithering characteristic F (under assumption of the continuous differentiability F on the real axis *R*) can be defined as a number $\sup_{x \in R} F'(x) - \inf_{x \in R} F'(x)$.

It can be easily seen that :

$$\sup_{x \in R} F'(x) - \inf_{x \in R} F'(x) = \sup_{x \in [-\Delta x/2, \Delta x/2]} F'(x) - \inf_{x \in [-\Delta x/2, \Delta x/2]} F'(x) = = \sup_{x \in [-\Delta x/2, \Delta x/2]} g'(x) - \inf_{x \in [-\Delta x/2, \Delta x/2]} g'(x)$$

where g is a periodic component of the dithering characteristic.

There are some easy to analyze particular cases of random variable *D* probability distributions.

Example 1. If dither is described with the Dirac measure concentrated in the point 0 then the dithering characteristic F is equal to a quantization function i.e. $F(x) = QN(x)$ for every $x \in R$. If dither is described with the Dirac measure concentrated in the point $b \in R$ then the dithering characteristic F is given for every $x \in R$ by the formula $F(x) = QN(x+b)$. As a result the dithering characteristic is a monotonic increasing function but is not continuous.■

Example 2. If a random variable D has a discrete distribution concentrated on a finite number of points then the dithering characteristic F is a monotonic function, constant inside intervals but not continuous. ■

Example 3. If a random variable D has a density f such that $\operatorname{supp} f$ is a subset of the closed interval $[-k_1\Delta x + b, k_2\Delta x + b]$, where $k_1, k_2 \in R^+$, $b \in R$ and $0 < k_1, k_2 < \frac{1}{2}$ then the dithering characteristic is continuous but constant on a number of intervals. Thus it is not invertible.■

Example 4. If a random variable D has a uniform distribution then dithering characteristic is continuous, monotonic increasing and linear inside intervals or linear. ■

Example 5. If D has a uniform distribution on the interval $[-\frac{1}{2}k\Delta x + b, \frac{1}{2}k\Delta x + b]$ for fixed $k \in N$ and $b \in R$ then the dithering characteristic is given for every $x \in R$ by the formula $F(x) = x + b$. In particular if D has a uniform distribution on the interval $[-\frac{1}{2}k\Delta x, \frac{1}{2}k\Delta x]$ for fixed $k \in N$ then the dithering characteristic is linear i.e. $F(x) = x$ for every $x \in R$. ■

Example 6. If D has a probability distribution f given by a formula

$$f(x) = \sum_{k=-\infty}^{+\infty} \alpha_k \chi_{[-\frac{1}{2}k\Delta x, \frac{1}{2}k\Delta x]}$$

(where $\alpha_k \in R^+$ and $\chi_{[-\frac{1}{2}k\Delta x, \frac{1}{2}k\Delta x]}$ is a characteristic function of the set $[-\frac{1}{2}k\Delta x, \frac{1}{2}k\Delta x]$) then the dithering characteristic is linear i.e. $F(x) = x$ for every $x \in R$. Then a uniform distribution on the interval $[-\frac{1}{2}k\Delta x, \frac{1}{2}k\Delta x]$ for fixed $k \in N$ is not a unique type of distribution giving a linear dithering characteristic. ■

Example 7. If D has a probability distribution f given by a formula

$$f(x) = \sum_{k=-\infty}^{+\infty} \alpha_k \chi_{[-\frac{1}{2}k\Delta x + b_k, \frac{1}{2}k\Delta x + b_k]}$$

where $\alpha_k \in R^+$, $b_k \in R$ and $\chi_{[-\frac{1}{2}k\Delta x + b_k, \frac{1}{2}k\Delta x + b_k]}$ is a characteristic function of the set $[-\frac{1}{2}k\Delta x + b_k, \frac{1}{2}k\Delta x + b_k]$ then there exists $b \in R$ that the dithering characteristic is given for every $x \in R$ by the formula $F(x) = x + b$. ■

It follows from our computer simulations the following, intuitively clear conclusion: "larger dither" gives a "more linear" dithering characteristic *F*. This rule does not work always but is useful from practical point of view.

On Fig. 2-4 some exemplary dithering characteristics are shown .

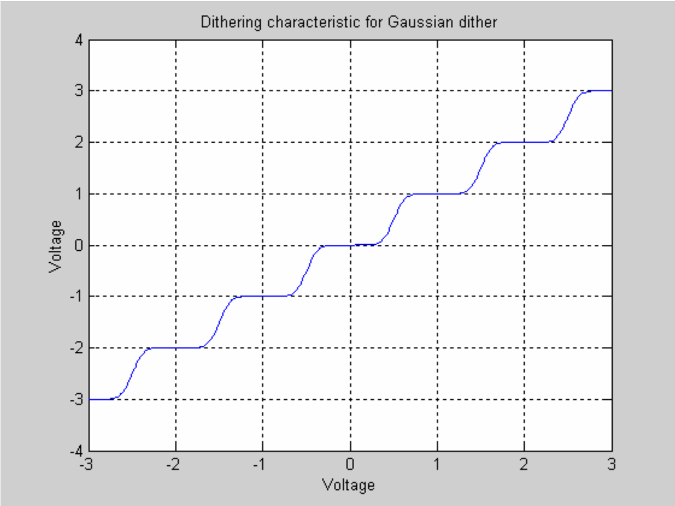

Fig. 2. Exemplary dithering characteristic when the dither distribution is Gaussian (with the mean value 0 and the standard deviation σ=0.1Δx) . Quantization function is uniform with Δx=1V.

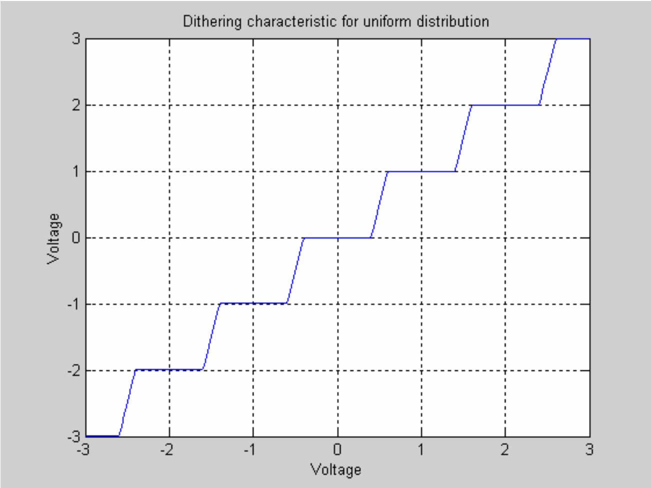

Fig. 3. Exemplary dithering characteristic when the dither distribution is uniform on the interval $[-0.1\Delta x, 0.1\Delta x]$. Quantization function is uniform with $\Delta x = 1\ V$.

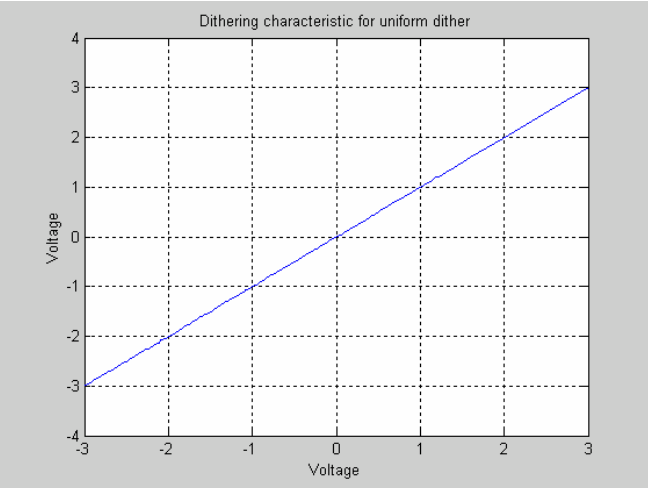

Fig. 4. Exemplary dithering characteristic when the dither distribution is uniform on the interval $[-\frac{1}{2}k\Delta x, \frac{1}{2}k\Delta x]$ for k=5.5. Quantization function is uniform with $\Delta x = 1\ V$.

3.GENERALIZED DITHERING METHOD

Assume that a random sequence $(D_n)_{n=1}^{\infty}$ is a sequence of independent (or stationary in the strict sense) random variables with the same one dimensional probability distribution as the random variable D. The sequence $(D_n)_{n=1}^{\infty}$ describes the process of dither addition as a sequence of independent (or stationary in the strict sense) experiments. We assume that $D = D_1$ and $D \in L^1(\Omega, \Sigma, P)$). In this case we have of course $D_n \in L^1(\Omega, \Sigma, P)$ for every $n \in N$. Hence for every $n \in N$ and every $a \in R$ we obtain $QN(a + D) \in L^1(\Omega, \Sigma, P)$ and $QN(a + D_n) \in L^1(\Omega, \Sigma, P)$.

Additionally assume, that the dithering characteristic $F : R \to R$ (see section 2) is continuous and strictly monotonic increasing. Hence $F : R \to R$ is invertible.

We can compute values of F with arbitrary accuracy computing values of the periodic component of the function F . To be exact we have to compute values only on the interval which has the length equal to the period of the periodic component .

The generalized dithering method is simple. We know the dithering characteristic $F : R \to R$ for a given distribution of the random variable D and we take N_0 values

$$QN(a + D_1)(\omega), QN(a + D_2)(\omega), ..., QN(a + D_{N_0})(\omega) \qquad (2)$$

where $\omega \in \Omega$ is an elementary event. Then we compute the mean value $\frac{1}{N_0}\sum_{n=1}^{N_0} QN(a + D_n)(\omega)$. A finite sequence (2) is composed of N_0 independent realizations of the random variable $QN(a + D)$ or in other words N_0 first coefficients of the trajectory of the stochastic process $(QN(a + D_n))_{n=1}^{\infty}$.

Using Strong Law of Large Numbers we obtain that P almost everywhere if $N_0 \to \infty$ then

$$\frac{1}{N_0}\sum_{n=1}^{N_0} QN(a + D_n)(\omega) \to E(QN(a + D) \qquad (3)$$

i.e. P almost everywhere if $N_0 \to \infty$ then we have

$$\frac{1}{N_0}\sum_{n=1}^{N_0} QN(a + D_n)(\omega) \to F(a) \qquad (4)$$

Because the function $F : R \to R$ (as strictly monotonic increasing and continuous) is invertible and an inverse function $F^{-1} : R \to R$ is continuous then from (4) we have P almost everywhere: if $N_0 \to \infty$ then

$$F^{-1}(\frac{1}{N_0}\sum_{n=1}^{N_0} QN(a + D_n)(\omega)) \to F^{-1}(F(a)) = a \qquad (5)$$

As an estimator of the value a we can admit

$$A(N_0) \overset{df}{=} F^{-1}(\frac{1}{N_0}\sum_{n=1}^{N_0} QN(a + D_n)) \qquad (6)$$

An estimate of the value a is then computed as a realization of the random variable $A(N_0)$ i.e.

$$A(N_0)(\omega) \overset{df}{=} F^{-1}(\frac{1}{N_0}\sum_{n=1}^{N_0} QN(a + D_n(\omega))) \qquad (7)$$

The formula (5) says that the estimator (7) is strong consistent . From the convergence P almost everywhere it follows the asymptotic convergence of random variables (i.e. convergence in probability). In our case it means that if $N_0 \to +\infty$ then the convergence P almost everywhere $A(N_0) \to a$ implies the convergence $A(N_0) \to a$ in probability. Then we can say: for every $\varepsilon > 0$ and every $\delta > 0$ there is such $\tilde{N}_0 \in N$ that for every $N_0 \geq \tilde{N}_0$ we have

$$P(|A(N_0) - a| \geq \varepsilon) \leq \delta \qquad (8)$$

In short, the answer about the value a is the following: a (a final result of the measurement) is in fact a realization of the random variable $A(N_0)$. $A(N_0)(\omega)$ is an estimate of the value a. It is exactly the same case as parameter estimation in mathematical statistics. The inequality (8) gives a good intuitive description of the "practical value" of the estimate $A(N_0)(\omega)$.

We can obtain the classical dithering method as a special case of the described above generalized dithering method.

If the random variable D describing dither has the uniform distribution on the interval $[-\frac{1}{2}k\Delta x, \frac{1}{2}k\Delta x]$ for fixed $k \in N$ then the dithering characteristic is linear i.e. $F(x) = x$ because under these assumptions $D \in L^1(\Omega, \mathcal{F}, P)$ and we have

$$E(QN(a + D) = a \tag{9}$$

In this situation we have not to compute the inverse of the function F. The formula (9) is the essence of the classical dithering method. The equality (9) can be understood in the following way: averaging of samples cancels nonlinearity of the quantization function QN in the formula (9).

Finally in the classical dithering method we take as $A(N_0)(\omega)$ (an estimate of the value a) the average i.e.

$$A(N_0)(\omega) = \frac{1}{N_0}\sum_{n=1}^{N_0} QN(a + D_n(\omega)) \tag{10}$$

Comment. All over the paper we assume that $(D_n)_{n=1}^{\infty}$ is a sequence of independent random variables with the same probability distribution. The dithering method works correctly also in the more general case when a sequence $(D_n)_{n=1}^{\infty}$ is a sequence of real random variables stationary in the strict sense. In both cases we assume that $(D_n)_{n=1}^{\infty}$ fulfills the additional condition: for every $n \in N$ we have $D_n \in L^1(\Omega, \mathcal{F}, P)$.

Asymmetry of the random variable D distribution causes errors in the classical dithering method. In generalized method it is not important because we correct these errors by inverting the dithering characteristic F. But in both dithering methods true information about a distribution P_D of the random variable D describing dither is a crucial point for accuracy.

When we do not use the true distribution of the random variable D (in dithering characteristic computations) it introduces usually a systematic error of the method.

Assume F_1 and F_2 are continuous and strictly monotonic increasing dithering characteristics. F_1 denotes a true dithering characteristic, and F_2 an admitted dithering characteristic.

For a systematic error of the method we can take a number $F_2^{-1}(F_1(a)) - F_1^{-1}(F_1(a)) = F_2^{-1}(F_1(a)) - a$ but as a rule a is unknown then more convenient solution is taking as a systematic error (denote it by δ_0) an assessment done in the following way

$$\delta_0 \le \sup_{x \in R}\left|F_2^{-1}(x) - F_1^{-1}(x)\right|.$$

We would like to prove the fact "if distributions of two random variables D^k and D describing dither are sufficiently "close" then dithering characteristic F_k and F (for D^k and D appropriately) are close too". The situation is explained by the following theorem".

Theorem 4.1 Assume a random variable $D \in L^1(\Omega, \mathcal{F}, P)$ has a probability density and for every $k \in N$, $D^k \in L^1(\Omega, \mathcal{F}, P)$ and a sequence $(D^k)_{k=1}^{\infty}$ is a uniform integrable family of random variables. If $D^k \to D$ converges weakly when $k \to +\infty$ then

1) for every $x \in R$, $F_k(x) \to F(x)$ when $k \to +\infty$ (point convergence), where F_k is a dithering characteristic for the random variable D^k and F is a dithering characteristic for the random variable D.

2) if for every $k \in N$ the random variable D^k has a density (related to the Lebesgue measure on the real axis) then $F_k \to F$ uniformly when $k \to +\infty$.

Proof. see [3] ■

Comment 1. Assumption of weak convergence is not an especially restrictive assumption. If $D^k \to D$ P almost everywhere when $k \to +\infty$, or $D^k \to D$ in probability or $D^k \to D$ in the norm of $L^p(\Omega, \mathcal{F}, P)$ (where $p \ge 1$) then $D^k \to D$ converges weakly when $k \to +\infty$.■

Comment 2. There are many natural examples of uniform integrability. For instance a sequence $(D^k)_{k=1}^{\infty}$ is a family of uniform integrable random variables if there is such a bounded interval $I \subseteq R$, that $P_{D^k}(I) = 1$ for every $k \in N$.

Another example, if $D^k \in L^p(\Omega, \mathcal{F}, P)$ where $p \ge 1$ and $D^k \to D$ converges in $L^p(\Omega, \mathcal{F}, P)$ when $k \to +\infty$ to a random variable $D \in L^p(\Omega, \mathcal{F}, P)$ then a family of random variables $(D^k)_{k=1}^{\infty}$ is uniform integrable.■

5.SPEED OF CONVERGENCE OF THE ESTIMATOR $A(N_0)$ TO THE MEASURED VALUE a

Assume the assumptions of the section 4 (concerning the random variable D which describes dither) are fulfilled. Assume additionally that, $D \in L^2(\Omega, \mathcal{F}, P)$ (equivalently we can say that there is a variance of the random variable D). From results of the section 2 we have $QN(a + D) \in L^2(\Omega, \mathcal{F}, P)$ and there is a variance $D^2(QN(a + D))$.

As a value of the measured voltage a we admit a number:

$$A(N_0)(\omega) \overset{df}{=} F^{-1}\left(\frac{1}{N_0}\sum_{n=1}^{N_0} QN(a + D_n(\omega))\right) \qquad (11)$$

Quality of the estimator $A(N_0)$ (i.e. its errors) can be assessed with the variance $D^2(A(N_0))$. As will be proved in the sequel the variance $D^2(A(N_0))$ can be assessed with a sequence which tends to 0 when $N_0 \to +\infty$. Therefore we can control the value of the variance (increasing number N_0) and taking N_0 suited to the needed accuracy.

As was proved in the section 4 it follows from Strong Law of Large Numbers that $A(N_0) \to a$ when $N_0 \to +\infty$ (P almost everywhere) and also $A(N_0) \to a$ (in probability) (which is in electronic measurement practice is more intuitive), see the inequality (4.7).

The inequality (8) does not tell us how to choose N_0 for an acceptable level of accuracy. Such a mechanism gives the Chebyshev inequality (see appendix). From the Chebyshev inequality we obtain that for an arbitrary $\varepsilon > 0$ we have

$$P(|A(N_0) - a| \geq \varepsilon) \leq \frac{D^2(A(N_0))}{\varepsilon^2} \qquad (12)$$

If we prove the convergence $D^2(A(N_0)) \to 0$ when $N_0 \to +\infty$ then we will be able to control accuracy of estimation by choosing appropriate N_0.

We can easily assess the variance $D^2(A(N_0))$ assuming that the dithering characteristic F is continuously differentiable on R and $F'(x) > 0$ for every $x \in R$ (then F is strictly monotonic increasing). Denote $A_0 \overset{df}{=} \inf_{t \in R} F'(t)$.

We have in this situation

$$A_0 \overset{df}{=} \inf_{t\in R} F'(t) = \inf_{t \in [-\Delta x/2, \Delta x/2]} F'(t) = 1 + \inf_{t \in [-\Delta x/2, \Delta x/2]} g'(t) > 0,$$

where g is a periodic component of the dithering characteristic. In the case of classical dithering method we have $A_0 = 1$.

$$D^2(A(N_0)) = D^2\left(F^{-1}\left(\frac{1}{N_0}\sum_{n=1}^{N_0} QN(a + D_n)\right)\right) \leq$$

$$\leq D^2\left(\frac{1}{A_0 N_0}\sum_{n=1}^{N_0} QN(a + D_n)\right) =$$

$$= \frac{1}{A_0^2 N_0^2} D^2\left(\sum_{n=1}^{N_0} QN(a + D_n)\right) = \frac{1}{A_0^2 N_0} D^2(QN(a + D_n)$$

Thus the assessment of the variance $D^2(A(N_0))$ is inversely proportional to N_0 and $D^2(A(N_0)) \to 0$ when $N_0 \to +\infty$. The obtained result can be formulated in this way: "accuracy of the dithering method is proportional to $\sqrt{N_0}$ ".

5. CONCLUSIONS

1. It is possible to generalize the classical dithering method for wide class of dither D probability distributions and quantization functions QN (we can use for instance a uniform quantization function or a two level quantizer with saturation).

2.A dithering characteristic F is very useful notion. It allows assessment of the accuracy of the classical and generalized dithering methods. In particular we can easily assess systematic errors of the methods.

3. Accuracy of generalized dithering methods depends on accuracy of the generated distribution of the random variable D . In practice real distribution of the random variable D can differ from admitted one.

4. Random changes of comparison levels of the quantizer can be treated as dithering and can be in natural way taken into account in the generalized dithering method.
But non accurate identification of the random variable L (describing fluctuations of the comparison level of the quantizer) can also influence accuracy of measurements.

5. Under natural assumptions on the dither D, the variance $D^2(A(N_0))$ of the estimator $A(N_0)$ (of the measured voltage a) is inversely proportional to N_0, where N_0 is a number of averaged samples.

6. From many computer simulations it follows an intuitively clear conclusion: "greater dither" gives "more linear" dithering characteristic F. This rule does not work in general but is useful from practical point of view.

REFERENCES

[1] R.A.Wannamaker, "Subtractive and Nonsubtractive Dithering: A Mathematical Comparison", J. Audio Eng. Soc., Vol. 52, No. 12, 2004, pp. 1211-1227.

[2] R. van de Plassche, "Integrated Analog-to_digital and Digital-to-Analog Converters", Kluwer, 1994.

[3] T.Adamski, "Dithering Methods in A/D Conversion for Uniform Quantizers and Errors Introduced by Dither", Electronics and Telecommunications Quarterly, Vol. 52, no. 1, 2006, pp 49-72.

[4] T.Adamski, "Generalized Dithering Methods for A/D Converters", Proceedings of the Polish-Czech-Hungarian Workshop on Circuit Theory, Signal Processing and

Applications; Prague, September, 2003, pp 123-127.