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INFLUENCE OF QUANTIZATION NOISE ON DFT–BASED DSP ALGORITHMS

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Abstract: This paper deals with analysis of influence of quantization noise on estimation of parameters of signal gained by various algorithms based on Discrete Fourier Transform (DFT) for the case of coherent and non–coherent sampling. Theoretical analysis, numerical simulation results and experimental validation are presented. The analyzed methods are compared from the point of view of their sensitivity to quantization noise.

Keywords: DFT, uncertainty analysis, quantization noise.

1. INTRODUCTION

All digital measuring instruments and systems work with digitized signals, i.e. signals discrete both in time and in magnitude. Sampling analog signal in time results in periodicity of signal frequency spectrum, signal amplitude quantization leads to corresponding component in uncertainty of measured signal. Qmtization noise is additive noise of input signal samples and is usually supposed to be uniformly distributed white noise, with standard deviation directly proportional to the quantization step of the used quantizer (analog-to-digital converter). Its mean value is zero in most cases (if rounding is used in A/D conversion). Since A/D conversion is present in all digital measuring instruments, quantization noise is always present in signal sampled to be processed by the DSP algorithm implemented in given instrument. That is why analysis of its influence to measured value uncertainty is a very important task in measurement. Many papers were therefore devoted to analysis of influence of various components of the digital measuring instrument structure on measurement uncertainty. The sampling converter was analyzed in [1], influence of ADC quantization, sampling jitter and microprocessor finite wordlength in basic basic FFT algorithms for coherent sampling and without signal windowing was analyzed in [2], [3], numerical method based on computer simulation was selected to analyze measurement of mean square value of input signal in [4], [5].

Spectrum analysis of digitized signal belongs to the most common tasks in digital signal processing. The DFT algorithm represents the basic DSP algorithm for the spectral analysis. Since the measured signal is most frequently sampled non-coherently, windows are used to suppress the leakage. Various algorithms can be used to obtain estimate of signal parameters from the DFT spectrum of windowed signal. This paper is focused on uncertainty analysis of some of these algorithms. Quantization noise is supposed to be the main source of uncertainty. Signal parameter estimates are affected besides quantization also by leakage. The influence of leakage on estimated parameters was discussed in [6].

2. DFT AND QUANTIZATION NOISE

Since the estimate of signal parameters is based on processing of DFT spectrum, uncertainty of DFT spectrum influences the resultant uncertainty of the estimate. For the case of non-coherent sampling, cosine windows are most frequently applied to the sampled signal. Cosine time window $w_{\cos}(n)$ is defined as

$$w_{\cos}(n) = \sum_{l=0}^{L} a_l \cos\left(\frac{2\pi}{N} nl\right), \quad n = 0,1,\ldots,N-1 \tag{1}$$

where L is the window order, N is window length and a_l are coefficients specific for each window.

Frequency spectrum of windowed signal can be computed using the DFT algorithm and is given by

$$X(k) = \sum_{n=0}^{N-1} x(n)\, w(n)\, e^{-\frac{2\pi n}{N}k} \tag{2}$$

where N is DFT length and $x(n)$ are input signal samples.

The samples are affected by quantization noise with standard deviation

$$u_n = \frac{V_r}{2^{ENOB}\sqrt{12}} \tag{3}$$

where V_r is the full–scale range of the used ADC and $ENOB$ is its effective number of bits.

This quantization noise causes uncertainty of module u_M of DFT spectral line $M(k) = |X(k)|$ [7],

$$u_M^2 = \frac{N}{2} u_n^2\, nnpg \tag{4}$$

where $nnpg$ is normalised noise power gain defined as [8]

$$nnpg = \frac{1}{N}\sum_{n=0}^{N-1}\left(w(n)\right)^2 \tag{5}$$

Phase uncertainty can be expressed as [7],

$$u_{\varphi(k)}^2 = \frac{u_n^2}{M^2(k)} \frac{N}{2} nnpg \tag{6}$$

3. METHODS OF PARAMETERS ESTIMATION

Various methods are used to obtain signal parameters of multi–frequency signals (frequency, amplitude and phase of individual harmonic or interharmonic components) from DFT spectrum in coherent and in non-coherent sampling. Sinusoidal input signal is supposed in the analysis presented here. Results of this analysis are valid for each sinusoidal component of multi–sinusoidal signal (signal composed of sum of sinusoidal components).

3.1. *Coherent sampling*

Windowing is not used in the case of coherent sampling of multi–harmonic signal. Amplitude of spectral component V_i of the measured signal in this case is directly proportional to the module of the corresponding DFT spectral line

$$\begin{aligned} V_i &= \frac{2}{N} M(k) \qquad k \neq 0 \\ V_0 &= \frac{1}{N} M(0) \qquad k = 0 \end{aligned} \tag{7}$$

where $M(k)$ is the module of DFT spectral line representing the i-th harmonic component and $M(0)$ represents the spectral line corresponding to the mean value of signal V_0.

The uncertainty due to quantization noise of these parameters can be found from

$$u^2(V_i) = 2\frac{u_n^2}{N} \qquad i \neq 0 \tag{8}$$

$$u^2(V_0) = \frac{u_n^2}{N} \tag{9}$$

The uncertainty of estimate of phase spectrum lines can be computed by (6) for $nnpg = 1$ (rectangular window).

3.2. *Non–coherent sampling*

In case of processing of non–coherently sampled signal (non-integer number of signal periods sampled), windowing is used to reduce the leakage effect. Various algorithms can be used for determining signal parameters from the DFT spectrum of windowed signal. Two of them are mentioned in the next sections.

3.2.1 *RMS value estimation based on signal energy in the window main lobe*

Estimate of RMS value of a harmonic spectrum component can be found [8] as

$$X_{RMS} = \sqrt{\frac{1}{N^2 \cdot nnpg}\left(\sum_{f_k=f_1}^{f_2} \left|M(f_k)\right|^2 + \sum_{f_k=-f_2}^{-f_1} \left|M(f_k)\right|^2\right)} \tag{10}$$

Here f_1 and f_2 are frequencies defining the main lobe of the window spectrum shifted to the signal harmonic component and containing $2L+1$ DFT amplitude spectrum lines. $M(f_k)$ are components of amplitude spectrum and $nnpg$ is normalized noise power gain (5).

Uncertainty analysis of this algorithm is complicated because of nonzero covariance between spectral lines in the main window lobe. The analysis including this covariance was presented in [7] and the resultant formula for uncertainty of RMS value is

$$u^2_{X_{RMS}} = \frac{u_n^2}{N} ENBW_0 \tag{11}$$

where $ENBW_0$ is equivalent-noise bandwidth of the squared window (equal to 1 for rectangular window) and given by [9]

$$ENBW_0 = N \frac{\sum_{n=0}^{N-1} w^4(n)}{\left(\sum_{n=0}^{N-1} w^2(n)\right)^2} \tag{12}$$

3.2.2 *Windowing and interpolation in frequency domain*

Another method is based on signal windowing and interpolation in frequency domain (W/IFD). Interpolation based on Rife–Vincent windows of the first class that we analyze below is described in [10]. This method is based on finding the "exact" frequency of each spectral component of the measured signal and on known window spectrum shape. Knowledge of frequency (in case of non-coherent sampling not placed on DFT grid) allows subsequent estimate of amplitude and phase of the investigated spectral component. The uncertainty analysis focused on the important special case of using Hann window is presented in [11].

Estimate of exact frequency

In the interpolation in frequency domain based on Rife–Vincent windows of the first class (RV1), finding the exact frequency is based on ratios of the local maximum value $M(k)$ to its neighboring values of the amplitude spectrum

$$\beta_1 = \frac{M(k+1)}{M(k)}, \quad \beta_2 = \frac{M(k-1)}{M(k)} \tag{13}$$

where $M(k)$, $M(k+1)$ and $M(k–1)$ are modules of the DFT spectrum of signal. Estimates of fractional frequency δ relative to frequency bin (see Fig.1) corresponding to ratios β_1 and β_2 are given by

$$\delta_1 = \frac{(L+1)\beta_1 - L}{1+\beta_1}, \quad \delta_2 = \frac{L-(L+1)\beta_2}{1+\beta_2} \tag{14}$$

where L is order of cosine window (see (1)). RV1 windows are a subset of cosine windows for the concrete coefficients values, see Tab. 5).

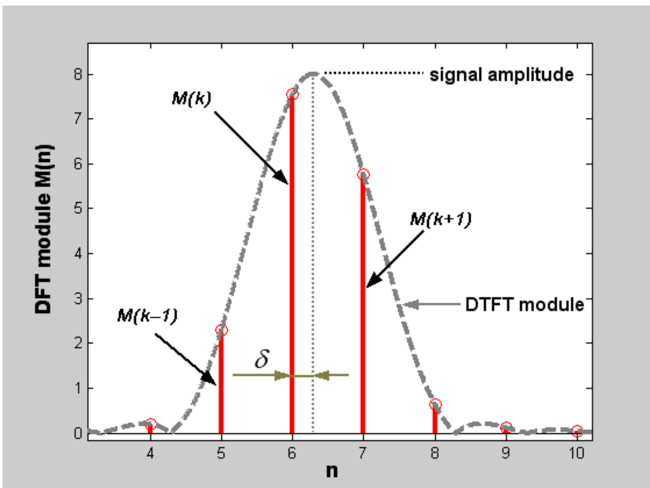

Fig.1 Principle of interpolation in frequency domain

Resultant estimate of frequency displacement (fractional frequency) δ is determined as average of δ_1 and δ_2

$$\delta = \frac{\delta_1 + \delta_2}{2} \tag{15}$$

Estimate of frequency of corresponding harmonic component is therefore

$$f = (k + \delta)\Delta f \tag{16}$$

where Δf is frequency bin ($\Delta f = f_s/N$), f_s is sampling frequency and N is length of DFT.

Estimate of amplitude and phase

Estimate of amplitude and phase of a spectral component is based on interpolation of signal DFT spectrum to find the approximate value of local maximum of Discrete-Time Fourier Transform (DTFT) spectrum. The DTFT spectrum is a continuous function of frequency, see Fig.1. Spectrum of the used window has to be known. Spectrum of a cosine window (1) is

$$W(\eta) = \frac{N\eta}{\pi} e^{-j\pi\frac{N-1}{N}\eta} \sin(\pi\eta) \cdot \sum_{l=0}^{L} \frac{a_l}{\eta^2 - l^2} \tag{17}$$

where η is real number from the interval <0, N >, N is length of the DFT and a_l are window coefficients.

Signal windowing in the time domain corresponds to the convolution of signal and window spectra in the frequency domain. Since sinusoidal signal FT spectrum is Dirac pulse placed on signal frequency, the above-mentioned convolution means only shifting window spectrum to the signal frequency. An estimate of amplitude of the investigated spectral component can be found for the known δ as

$$A = \frac{2\pi}{N} \frac{M(k)}{|D|} \tag{18}$$

where

$$D = \gamma_0(\delta)\delta \sin(\pi\delta) \tag{19}$$

and

$$\gamma_0(\delta) = \sum_{l=0}^{L} \frac{a_l}{\delta^2 - l^2} \tag{20}$$

The estimate of phase is based on the linearity of phase of cosine windows. For the known δ it can be obtained as

$$\varphi = angle(X(k)) - \pi \frac{N-1}{N}\delta + \frac{\pi}{2} sign(D) \tag{21}$$

where $angle(X(k))$ is phase of the DFT spectrum (2). This expression represents the phase related to the beginning of sampling. If we are interested in phase difference only, this difference can be found as the difference of phase values of the windowed DFT spectra (spectral leakage has the same effect on phase values of both signals having the same frequency).

Uncertainty analysis

Since the interpolated DFT spectrum is affected by quantization noise, signal parameters based on this spectrum are as well corrupted by quantization noise. Estimate of uncertainty due to quantization can be derived by using the law of propagation of uncertainties [12].

Uncertainty of frequency

All estimates of parameters depend on fractional frequency bin δ. Therefore also uncertainty of estimates of these parameters depend on uncertainty of δ. That is why it is necessary to find firstly the uncertainty of fractional frequency bin δ.

As can be seen from (13) – (15), δ depends on some ratios of modules of DFT. We shall therefore firstly find uncertainty of ratios β_1 and β_2. From the law of propagation of uncertainties there is

$$u^2(\beta_i) = \frac{u_M^2}{M^2(k)}\left(1 - 2\beta_i r_1 + \beta_i^2\right) \tag{22}$$

where i = 1, 2, u_M is uncertainty of the DFT module (4) and r_1 is correlation coefficient between neighboring two modules of DFT spectrum ($M(j)$ and $M(j+1)$). The r_1 can be expressed as [7]

$$r_1 = \frac{\left| a_0 a_1 + \frac{1}{2}\sum_{n=1}^{L-1} a_n a_{n+1} \right|}{nnpg} \tag{23}$$

Knowing uncertainty of β_1 and β_2, it is possible to derive uncertainty of δ_1 and δ_2

$$u^2(\delta_1) = \left(\frac{2L+1}{(1+\beta_1)^2}\right)^2 u^2(\beta_1) \tag{24}$$

$$u^2(\delta_2) = \left(-\frac{2L+1}{(1+\beta_2)^2}\right)^2 u^2(\beta_2) \tag{25}$$

Here L is the cosine window order (1).

To obtain resultant formula for uncertainty of δ it is necessary take into account the covariance between δ_1 and δ_2. These values are dependent on β_1 and β_2 that are determined by DFT modules (13). Since we know the covariance of these modules, it is convenient firstly to derive the correlation coefficient between these quantities

$$r(\beta_1, \beta_2) = \frac{\beta_1\beta_2 - (\beta_1 + \beta_2)r_1 + r_2}{\sqrt{(1 - 2\beta_1 r_1 + \beta_1^2)(1 - 2\beta_2 r_1 + \beta_2^2)}} \tag{26}$$

Here r_2 is correlation coefficient between two modules of DFT spectrum $M(j-1)$ and $M(j+1)$. It can be expressed as [7]

$$r_2 = \frac{\left| a_0 a_2 + \frac{a_1^2}{4} + \frac{1}{2}\sum_{n=1}^{L-2} a_n a_{n+2} \right|}{nnpg} \tag{27}$$

It can be shown that the demanded correlation coefficient between δ_1 and δ_2 is

$$r(\delta_1, \delta_2) = -r(\beta_1, \beta_2) \tag{28}$$

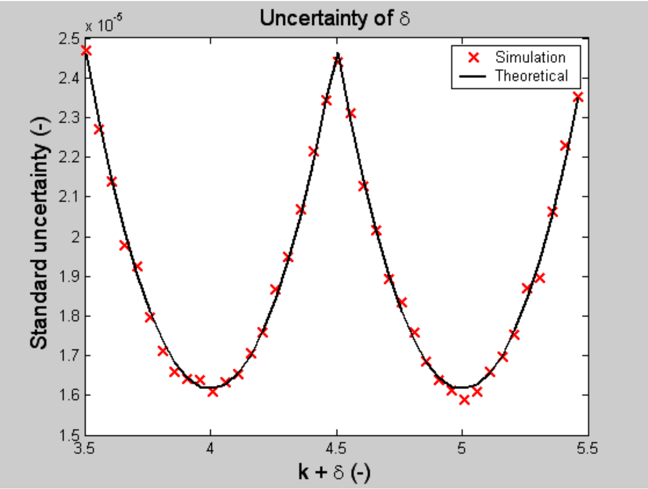

Fig.2 Uncertainty of fractional frequency bin δ as a function of normalized frequency, Hann window used

Based on previous formulae the resultant uncertainty of δ can be expresses as

$$u(\delta)=\frac{1}{2}\sqrt{u^2(\delta_1)+u^2(\delta_2)+2u(\delta_1)u(\delta_2)r(\delta_1,\delta_2)} \tag{29}$$

Fig.2 shows simulation results and theoretical values of standard uncertainty of δ due to quantization (sinusoidal signal, V_{pp} = 9.9 V, zero mean value, DFT length N = 512, ADC full-scale range V_r = 10 V and its effective number of bits $ENOB$ = 11.2, 10000 repetitions, not including bias of δ caused by sidelobes of spectrum corresponding to mirrored spectral component). Quantization noise was modeled by uniformly distributed additive noise.

The uncertainty of frequency estimation of signal is according to (16)

$$u(f)=u(\delta)\cdot\Delta f \tag{30}$$

Uncertainty of signal component amplitude

Uncertainty of amplitude of sinusoidal component of signal can be found by applying the law of uncertainty propagation on equation (18). We get

$$u^2(A)=\left(\frac{2\pi}{N}\right)^2\left[\left(\frac{\partial A}{\partial M(k)}\right)^2 u_M^2+\left(\frac{\partial A}{\partial \delta}\right)^2 u^2(\delta)+ +2\frac{\partial A}{\partial M(k)}\frac{\partial A}{\partial \delta}u(M(k),\delta)\right] \tag{31}$$

where $u(M(k),\delta)$ is the covariance between $M(k)$ and δ and it can be expressed as

$$u(M(k),\delta)=\frac{u_M^2}{2}\frac{(2L+1)}{M(k)}\left(\frac{\beta_2-r_1}{(1+\beta_2)^2}-\frac{\beta_1-r_1}{(1+\beta_1)^2}\right) \tag{32}$$

After expressing derivatives in (11), the resultant uncertainty of amplitude can be written as

$$u^2(A)=\left(\frac{2\pi}{N}\right)^2 A^2\left(\left(\frac{1}{M(k)}\right)^2 u_M^2+ +\left(\frac{C}{|D|}\right)^2 u^2(\delta)-2\frac{1}{M(k)}\frac{C}{|D|}u(M(k),\delta)\right) \tag{32}$$

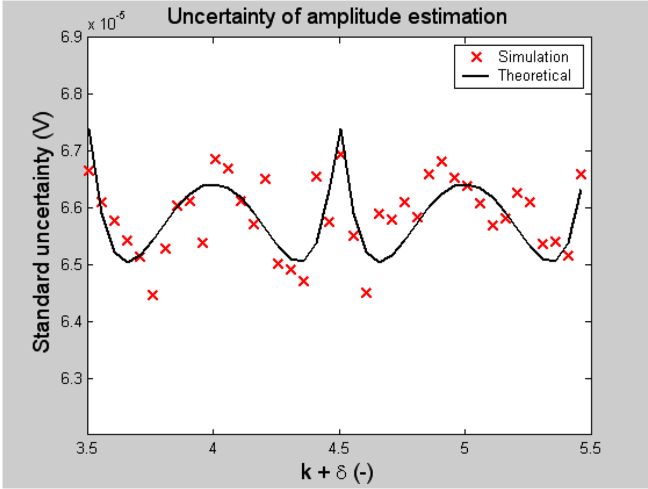

Fig.3 Uncertainty of signal amplitude estimation as a function of normalized frequency, Hann window used

where

$$C=\frac{|D|}{\delta}+\frac{|D|}{\cos(\pi\delta)}-2\delta^2\sin(\pi\delta)\sum_{l=0}^{L}\frac{a_l}{\left(\delta^2-l^2\right)^2} \tag{33}$$

and D is defined by the equation (19).

Dependence of uncertainty of amplitude estimation on normalized frequency is depicted in Fig. 3.

Uncertainty of phase

Uncertainty of phase is according to (21) determined by uncertainty of DFT phase spectrum (6) and uncertainty of δ (29)

$$u^2(\varphi)=u_{\varphi(k)}^2+\left(\frac{N-1}{N}\pi\right)^2 u^2(\delta) \tag{34}$$

where $u_{\varphi(k)}$ is uncertainty of the DFT phase spectrum (6) and $u(\delta)$ is uncertainty of fractional bin δ (29). The fractional frequency bin δ depends on the module of DFT spectrum. The mutual covariance between fractional bin δ and the phase spectrum was experimentally found to be negligible.

Dependence of uncertainty of phase estimation on normalized frequency is depicted in Fig. 4.

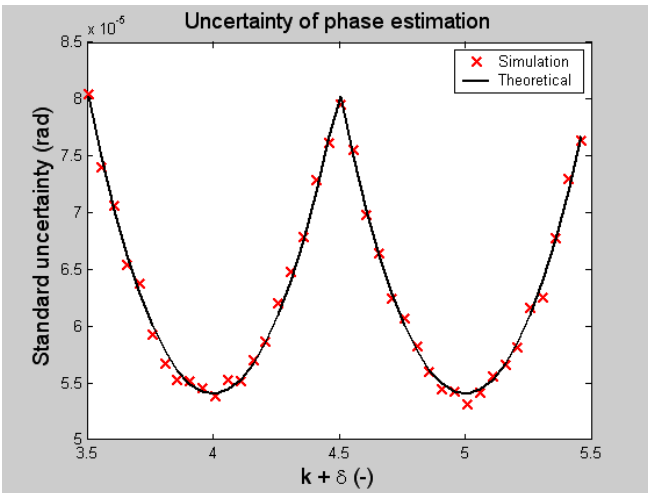

Fig.4 Uncertainty of signal phase estimation as a function of normalized frequency, Hann window used

4. COMPARISON OF THE DESCRIBED METHODS

Using windows leads to the leakage reduction, however it influences also the uncertainty of resultant parameter due to quantization. Theoretical values of uncertainty were verified by measurements. Tab. 1 to Tab. 4 allow comparison of presented theory and measurements. The HP3245A generator was used for generation of a sinusoidal with defined magnitude and frequency. Signal was digitised by a low-cost multiplexed 12-bit 200 kS/s DAQ plug-in board (6023E of National Instruments). Instruments were controlled by GPIB, MATLAB toolboxes Data Acquisition and Instrument Control were used.

Measurement conditions were sinusoidal signal, $V_{pp} = 9.9$ V, zero mean value, DFT length $N = 512$, sampling frequency 20 kHz, ADC full-scale range $V_r = 10$ V and its effective number of bits $ENOB = 11.2$. Uncertainty was evaluated as standard deviation of 1000 measurements, bias of δ caused by sidelobes of spectrum corresponding to mirrored spectral component was not taken into account, because it is not caused by quntization noise. Signal frequency was selected so that $k+\delta$ =5.4 (6 for the case of coherent sampling). It was found that for the case of non-coherent sampling, uncertainty of the RMS value by using rectangular window can increase more than 10-times. When using other windows, the changes of uncertainty with δ is less than 10 %.

Table 1. Standard uncertainty of RMS value in case of coherent sampling (par 3.1).

Window type	Standard uncertainty (V)	
	theoretical	*measured*
Rectangular (no window)	$5.9 \cdot 10^{-5}$	$5.9 \cdot 10^{-5}$

Table 2. Standard uncertainty of RMS value computed from window spectrum main lobe (par 3.2.1).

Window type	Window order *L*	Standard uncertainty (V)	
		theor.	*meas.*
Hann	1	$7.6 \cdot 10^{-5}$	$7.6 \cdot 10^{-5}$
Hamming	1	$7.3 \cdot 10^{-5}$	$7.4 \cdot 10^{-5}$
Blackman	2	$8.3 \cdot 10^{-5}$	$8.4 \cdot 10^{-5}$
7 Terms Blackman–Harris	3	$9.0 \cdot 10^{-5}$	$9.1 \cdot 10^{-5}$

Table 3. Standard uncertainty of RMS value computed by means of W/IFD method (par 3.2.2).

Window type	Window order *L*	Standard uncertainty (V)	
		theoretical	*measured*
RV, class 1	1 (Hann)	$2.2 \cdot 10^{-5}$	$2.5 \cdot 10^{-5}$
RV, class 1	2	$2.6 \cdot 10^{-5}$	$2.8 \cdot 10^{-5}$
RV, class 1	3	$3.0 \cdot 10^{-5}$	$3.2 \cdot 10^{-5}$
RV, class 1	4	$3.4 \cdot 10^{-5}$	$3.6 \cdot 10^{-5}$

Table 4. Standard uncertainty of fractional frequency bin δ computed by means of W/IFD method (par 3.2.2).

Window type	Window order *L*	Standard uncertainty (-)	
		theoretical	*measured*
RV, class 1	1 (Hann)	$6.6 \cdot 10^{-5}$	$7.4 \cdot 10^{-5}$
RV, class 1	2	$7.4 \cdot 10^{-5}$	$7.6 \cdot 10^{-5}$
RV, class 1	3	$8.1 \cdot 10^{-5}$	$8.1 \cdot 10^{-5}$
RV, class 1	4	$8.6 \cdot 10^{-5}$	$8.7 \cdot 10^{-5}$

Table 5. Coefficients of windows Rife-Vincent, class 1, for window order *L*

L	Rife–Vincent window coefficients				
	a_0	$-a_1$	a_2	$-a_3$	a_4
1	1/2	1/2	--	--	--
2	3/8	4/8	1/8	--	--
3	10/32	15/32	6/32	1/32	--
4	35/128	56/128	28/128	8/128	1/128

Tab. 5 presents values of coefficients of windows Rife-Vincent, class 1, for window orders up to four. Their values are taken over from [10] but changed so that the maximum value of the window is equal to 1.

5. CONCLUSION

Windowing suppresses leakage on the one hand, but increases uncertainty of estimated parameter due to quantization (compared with coherent sampling) on the other hand. The higher order of window, the higher leakage suppression, but also the higher uncertainty due to quantization. However, as can be seen from the tables, the uncertainty increases relatively slightly with window order increasing. Uncertainties analyzed in this paper represent the uncertainty component due to quantization noise only. The uncertainty component caused by long-range leakage from other signal components (e.g. the mirrored component in case of sinusoidal input signal) were mentioned in [6].

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REFERENCES

[1] J. Jakubiec, K. Konopka, ”Uncertainty propagation model of A/D measuring chain”, Proc. of 10th international IMEKO TC-4 Symposium Naples, Italy, September 1998, pp. 831-836.

[2] G. Betta, C. Liguori, A. Pietrosanto, ”Uncertainty analysis in fast Fourier Algorithms”, Proc. of 10th international IMEKO TC-4 Symposium Naples, Italy, September 1998, pp.747-752.

[3] G. Betta, C. Liguori, A. Pietrosanto, ”Propagation of uncertainty in a discrete Fourier transform algorithm”, Measurement, vol. 27, 2000, pp. 231-239

[4] N. Locci, C. Muscas, E. Ghiani, ”Evaluation of uncertainty in digital processing of quantized data”, Proc. of 11th international IMEKO TC-4 Symposium Lisbon, Portugal, September 2001, pp.102-106.

[5] N. Locci, C. Muscas, E. Ghiani, ”Evaluation of uncertainty in measurements based on digitized data”, Measurement, vol. 32, 2002, pp. 265-272

[6] M. Novotny, M. Sedlacek, “Measurement of RMS values of non-coherently sampled signals”, Proc. of 13th international IMEKO TC-4 Symposium Athens, Greece, October 2004, pp.230-235.

[7] M. Novotný, D. Slepička, M. Sedláček, “Uncertainty analysis of the RMS value and phase”, not published

[8] O. M. Salomon, “The Use of DFT Windows on SNR and Harmonic Distortion Computations”, IEEE Transactions on Instrumentation and Measurement, vol. 43, no. 2, pp. 194-199, April 1994

[9] P. Carbone, E. Nunzi, D. Petri, ”Windows for ADC Dynamic Testing via Frequency–Domain Analysis”, IEEE Transaction on Instrumenatation and Measurement, vol. 50, no.6, December 2001, pp.1571-1575, August 1989.

[10] G. Andria, M. Savino, A. Trotta, ”Windows and interpolation algorithms to improve electrical measurement accuracy”, IEEE Transactions on Instrumentation and Measurement, vol. 38, no. 4, pp. 856-863, August 1989.

[11] D. Agrež, ”Propagation of Uncertainty in the Interpolated DFT”, Proc. of IMTC 2003, Vail, CO, USA, May 2003, pp. 1148-1153.

[12] Guide to the Expression of Uncertainty in Measurement, International Organization for Standardization, Geneve, Switzerland, 1993