XVIII IMEKO WORLD CONGRESS
Metrology for a Sustainable Development
September, 17 – 22, 2006, Rio de Janeiro, Brazil

A NEW DISTANCE FOR FUZZY DESCRIPTIONS OF MEASUREMENTS

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Abstract - Fuzzy nominal scales were introduced in order to propose a formalism to the representation of empirical quantities by fuzzy subsets of words. This paper presents the results of studies on distances associated to this formalism and proposes a new distance operator.

Keywords: Measurement theory, Fuzzy description, Fuzzy nominal scale, Fuzzy distance.

1. INTRODUCTION

The introduction of the fuzzy subset theory in the measurement field takes its origin in 1971 in the Zadeh's paper [1] that exposes a mechanism of description of a quantity by a fuzzy subset of symbols. Since this paper, the definition of the description process was mainly based on good practices. Most of description processes had useful properties but it was sometime difficult to justify them. Recently, the link between the quantities and their fuzzy representation was formalized in a scale approach named the fuzzy nominal scale [2][3].

This approach gives the set of relations and operators that can be used to define equations on symbols such that these equations have a meaning on the set of quantity values. When a fuzzy description is a fuzzy nominal scale, a fuzzy equivalence relation on quantity values is linked to a fuzzy equivalence relation on their representation. This last relation also named similarity relation is used to define a distance between the fuzzy subset of symbols that represent the quantity values. This distance had been used to perform signal processing [4], but is useless to compare symbols that are not related by the similarity relation.

The purpose of this paper is to propose a distance on fuzzy representations linked to a metric on the set of quantity values. With this new distance, a fuzzy nominal scale is enhanced and the set of authorized operators has now this distance as member.

2. THE FUZZY SYMBOLISMS

The link between a physical state and its linguistic representation is characterized by a symbolism defined by the triplet $<E,S,R>$ where E is the set of physical states, S is the lexical set used to represent measurement results and R is a relation on ExS. Two mappings can be extracted from this relation: The *description mapping* denoted D associates a subset of S to any item of E, and the *meaning mapping* denoted M associates a subset of E to any item of S. These two mappings are linked with the following equation.

$$\forall e \in E, \forall s \in S, e \in M(s) \Leftrightarrow s \in D(e) \quad (1)$$

The R relation can be a fuzzy relation. Then, the translation of a physical state into its linguistic representation is called a *fuzzy linguistic description mapping* or simply a *fuzzy description mapping*. It transforms an object e of the set of physical states E into a fuzzy subset of linguistic terms called the *fuzzy description* of x. The dual mapping, called the *fuzzy meaning* mapping, associates a fuzzy subset of E to each term s of the lexical Set S. This fuzzy subset is the *fuzzy meaning* of s. In the paper the fuzzy subsets of linguistic terms also named lexical fuzzy subsets are denoted LFS. This two mappings are also linked:

$$\forall (e, s) \in E \times S, \mu_{M(s)}(e) = \mu_{D(e)}(s) \quad (2)$$

In [5] it is defined that $<E,S,R>$ is a ϕ-symbolism if the set of the meanings of the elements of S is a ϕ-partition of E and if each meaning is normalized. This paper restricts its investigation to id-symbolisms based on id-partition i.e. on Ruspini partition. The set of all possible LFS obtained by a fuzzy description based on id-symbolism is denoted $F_{id}(S)$. Any LFS respects then the condition:

$$\forall A \in F_{id}(S), \sum_{s \in S} \mu_A(s) = 1 \quad (3)$$

A fuzzy equivalence relation on the physical states can be associated to any id-symbolism.

$$\forall (x, y) \in E^2, \mu_{\sim}(x, y) = \sum_{s \in S} min(\mu_{M(s)}(x), \mu_{M(s)}(y)) \quad (4)$$

From this fuzzy equivalence relation and from the relation R, the following relation can be simply defined.

$$\forall (A, B) \in F_{id}(S)^2, \mu_{\sim}(A, B) = \sum_{s \in S} min(\mu_A(s), \mu_B(s)) \quad (5)$$

The symbolism $<E,S,R>$ is then considered as a fuzzy nominal scale.

3. CHOICE OF A DISTANCE OPERATOR

The relation used in the id-symbolism can define a distance between LFSs [4].

$$d_{\sim}(A, B) = 1 - \mu_{\sim}(A, B) \quad (6)$$

This distance is discriminant for LFSs that are at least partially equivalent but is equal to 1 when 2 LFS have empty intersection. This result is consistent with the absence of distance on the lexical set. This mean that the definition of a metric on the set $F_{Id}(S)$ needs the definition of a metric on the set S. Let d_S be a distance defined on S.

3.1. Required properties

The fuzzy subset theory proposes a large set of distance operators and the best way to select a distance operator is to list the properties that must be verified.

- The first property is the singleton coincidence: If two LFSs are singleton $\{s_1\}$ and $\{s_2\}$ then the distance between them is equal to the distance between symbols s_1 and s_2. This property supposes that the distance d_S on S exists.
- The continuity property is verified when the distance is a continuous mapping from SxS to the set of positive numbers.
- The precision property simply imposes that the distance between two LFSs must be a positive real number, and not a fuzzy subset of positive real numbers.
- The consistency property is usually verified by distances on crisp subsets: If A,B,C,E are four subsets of a metric space, d is the distance on this space, and d_g is a distance that generalize d on subsets, it verifies:

$$\begin{aligned}&\sup_{u \in A, v \in B} d(u, v) \le \inf_{x \in C, y \in E} d(x, y)\\ &\Rightarrow d_g(A, B) \le d_g(C, E)\end{aligned} \tag{7}$$

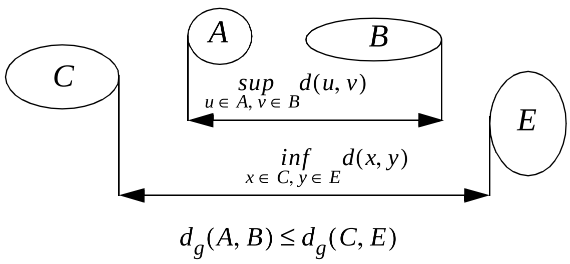

Fig. 1. Consistency property

The extension of a distance defined on a finite space to a distance defined on the set of the fuzzy subsets of this space was widely studied but, as shown below, no existing distance can be applied to $F_{Id}(S)$.

Distances on fuzzy subsets can by classified into the following categories.

- The distances that generalise an existing distance.
- The distances defined from a similarity measure.
- The distances defined with subset operators.
- The distances computed from a symbolic approach.

In our approach, a distance d_S is supposed to be defined on S. Then only the first cathegory is investigated.

The generalisation of a distance d_S defined on a finite set S, to a distance $d_{F(S)}$ defined on the set of fuzzy subsets of S is a recurrent subject of study. In [6] Bloch proposes four types of generalisation.

3.2. The geometrical approach

In this approach, fuzzy subsets in a ndimentional space are considered as crisp subsets in a (n+1)dimentional space. This means that the distance between membership degrees has the same semantic than distance on the ndimentional space. Such hypothesis can not be justified in our problem and this approach is not kept.

3.3. The fuzzification approach

In an other approach a distance D_S between crisp subsets is defined from the distance d_S. Then the distance D_S is fuzzyfied. In [7] three fuzzyfications of the Hausdorff distance are proposed.

$$H^1_{F_{Id}(S)}(F, G) = \int_0^1 H_S(F_\alpha, G_\alpha) d\alpha \tag{8}$$

$$H^\infty_{F_{Id}(S)}(F, G) = \sup_{\alpha \ge 0} H_S(F_\alpha, G_\alpha) \tag{9}$$

$$H^*_{F_{Id}(S)}(F, G) = H_S(F_{1.0}, G_{1.0}) \tag{10}$$

Where F_α and G_α are the alpha-cuts of F and G, and H_S is the Hausdorff distance:

$$H_S(A, B) = max(\max_{a \in A} \min_{b \in B} d_S(a, b), \max_{b \in B} \min_{a \in A} d_S(a, b)) \tag{11}$$

$H^\infty_{F_{Id}(S)}$ and $H^*_{F_{Id}(S)}$ do not verify the continuity property, but $H^1_{F_{Id}(S)}$ does. It also verifies the singleton coincidence, but not the consistency property.

3.4. The weighting approach

The distance d_S can be generalized with a weighting of membership degrees.

$$d_T(F, G) = \sum_{s_1 \in S} \sum_{s_2 \in S} T(\mu_F(s_1), \mu_G(s_2)) d_S(s_1, s_2) \tag{12}$$

Where T is a continuous triangular norm.

Such operator does not respect the separation axiom that imposes:

$$d_T(F, G) = 0 \Leftrightarrow F = G$$

then it cannot be considered as a distance.

3.5. A new approach

So a new distance that respects the four properties had been created. This distance is named the transportation distance d_{tp}. Its calculation is equivalent to the solution of a mass transportation problem [8]. It is similar to the Wasserstein distance used in probability theory, and can also be considered as a fuzzy version of the Levenshtein distance used to compare strings [10].

4. THE TRANSPORTATION DISTANCE

The transportation distance between two LFSs is based on the cost calculation of a set of transformations needed to transform the first LSF to the other. First a family of transformation mappings is defined:

Let $T_{s_i, s_j, x}$ be a mapping on a set $F_{id}(S)$ such that:

$$G = T_{s_i, s_j, x}(F) \Leftrightarrow \begin{array}{l} \mu_G(s_i) = \mu_F(s_i) - x \\ \mu_G(s_j) = \mu_F(s_j) + x \end{array} \tag{13}$$

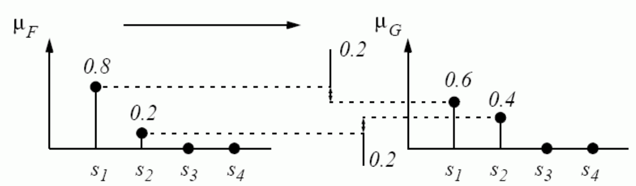

Fig. 2. The mapping $T_{s_1, s_2, 0.2}$ with $S=\{s_1,s_2,s_3,s_4\}$.

We demonstrate that any element of $F_{id}(S)$ can be transformed into any other element of $F_{id}(S)$ with the use of a sequence of such transformation mapping.

Let the following sets:

$$S_{F>G} = \{s \in S, \mu_F(s) > \mu_G(s)\}$$

$$S_{F=G} = \{s \in S, \mu_F(s) = \mu_G(s)\}$$

Proposition 1: Let S be a finite set. Let F and G be 2 elements of the set $F_{id}(S)$. Let $\Delta_s = \mu_F(s) - \mu_G(s)$. The following equality is verified:

$$\sum_{s \in S_{F>G}} \Delta_s = -\sum_{s \in S_{G>F}} \Delta_s \tag{14}$$

Then the definition of the sequence of transformation mapping is equivalent to the well known linear programming problem named the transportation problem [8]. The problem is to bring a product from a set of n_1 sources to a set of n_2 destinations. Each source i gives a quantity s_i of product, and each destination receives a quantity s_j of product. The total given quantity must be equal to the total received quantity:

$$\sum_{i=1}^{n_1} s_i = -\sum_{j=1}^{n_2} s_j' \tag{15}$$

A solution is to associate to each displacement from a source i to a destination j a quantity x_{ij} of transported product and a displacement unity cost c_{ij}. The aim is to find a solution that minimises the total cost:

$$\sum_{i,j} x_{\mathrm{ij}} c_{ij} \tag{16}$$

Considering the membership degrees as the transported product, the set $S_{F>G}$ as the set of sources and the set $S_{F<G}$ as the set of destinations, a distance can be computed as the total cost of the optimal solution for the transportation of membership degrees.

The transportation distance d_{tp} is defined on $F_{id}(S)$ from distance d_S on the lexical set S. The distance d_{tp} is the sum of the costs of each transformation mapping. And the cost of a transformation mapping $T_{s_i, s_j, x}$ is equal to: $x \cdot d_S(s_i, s_j)$.

It is now shown that the transportation distance is a distance, and it verifies the 4 constraints presented before.

For any $F, G, H \in F_{Id}(S)$, d_{tp} must verify:

$$d_{\mathrm{tp}}(F, G) = 0 \Leftrightarrow F = G, \tag{17}$$

$$d_{\mathrm{tp}}(F, G) = d_{\mathrm{tp}}(G, F), \tag{18}$$

$$d_{\mathrm{tp}}(F, G) + d_{\mathrm{tp}}(G, H) \geq d_{\mathrm{tp}}(F, H). \tag{19}$$

- The relation $F = G$ is equivalent to $S_{F>G} = S_{F<G} = \varnothing$ that is equivalent to $d_{\mathrm{tp}}(F, G) = 0$.
- The symmetry of d_{tp} is deduced from the symmetry of the transportation problem.
- Finally, $d_{\mathrm{tp}}(F, H)$ is by definition the distance corresponding to the optimal sequence of transformations $T_{s_i, s_j, x}$ that changes F into H. Then Adding a constraint in order to include G in the set of transformation steps will increase the distance.
- Calculating the distance between singletons $\{l_i\}$ and $\{l_j\}$ using the transportation problem is equivalent to finding the cheaper solution to bring a unity quantity of product from source i to destination j. The solution is made of only one transformation mapping $T_{s_i, s_j, 1}$. The cost of this solution is c_{ij} that is equal to the distance $d_S(s_i, s_j)$ then the singleton coincidence is verified.
- The precision and the continuity properties are deduced from the definition of the distance
- The consistency property of d_{tp} is demonstrated below:

Let F, G, H, I four elements of $F_{Id}(S)$ and $(s_1, s_2, s_3, s_4) \in S^4$ such that:

$$\overline{d_{FG}} \leq \underline{d_{HI}}. \tag{20}$$

where:

$$\underline{d_{HI}} = \inf_{(s_3, s_4) \in \mathrm{supp}(H) \times \mathrm{supp}(I)} d_S(s_3, s_4),$$

$$\overline{d_{FG}} = \sup_{(s_1, s_2) \in \mathrm{supp}(F) \times \mathrm{supp}(G)} d_S(s_1, s_2). \tag{21}$$

and $supp(A)$ is the support of A i.e. the set of lexical terms s such that $\mu_A(s) \neq 0$.

It must be shown that

$$d_{\mathrm{tp}}(F, G) \leq d_{\mathrm{tp}}(H, I). \tag{22}$$

If $H \cap I \neq \varnothing$, then $d_{HI} = 0$ and $\overline{d_{FG}} = 0$. F and G are the same singleton and $d_{\text{tp}}(F, G) = 0$. (22) is trivially verified.

If $H \cap I = \varnothing$, The brought quantity associated to the calculation of $d_{\text{tp}}(H, I)$ is equal to 1 and $d_{\text{tp}}(H, I) \geq d_{HI}$ because the transportation distance is then a weighted average of distances that are greater or equal to d_{HI}. With the same reasoning, $d_{\text{tp}}(F, G) \leq \overline{d_{FG}}$. Then, (20) induces $d_{\text{tp}}(F, G) \leq d_{\text{tp}}(H, I)$.

5. RESULTS

In this section, the transportation distance is applied on the hand posture recognition. More details on the application can be found in [9].

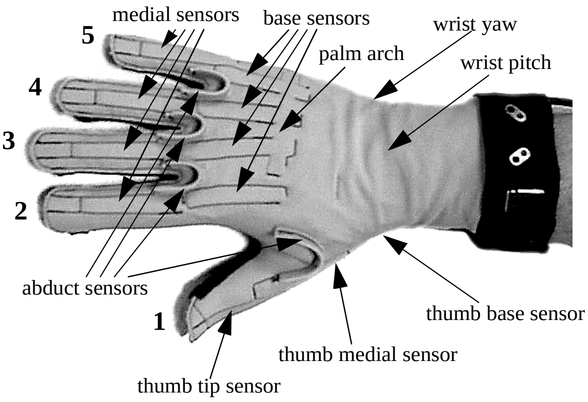

Fig. 3. : The 18 sensors of the Cyberglove®.

The hand posture is acquired with the 18 angle sensor of a CyberGlove® (Fig. 3.). The finger flexion (except for the thumb) is aquired with two angle sensors : MCP (métacarpal-phalanx angle) et IP (interphalanx angle). The linguistic description of a finger uses the set $S_{flexion}$ = {*Folded*, *Claw*, *Round*, *Square*, *Straight*} see Fig. 4.

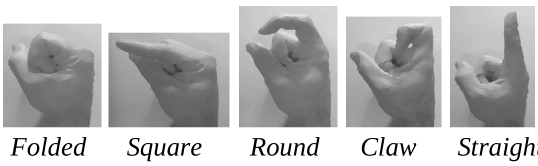

Fig. 4. Words used to describe the finger flexion.

The dataglove gives a numeric representation of the finger flexion as a couple $(\text{mcp}, \text{ip}) \in \Re^2$. The definition of the fuzzy linguistic description is performed through the definition of the fuzzy meaning of each lexical term. This meanings are fuzzy subsets in $\Re^2$ as shown in example in Fig. 5..

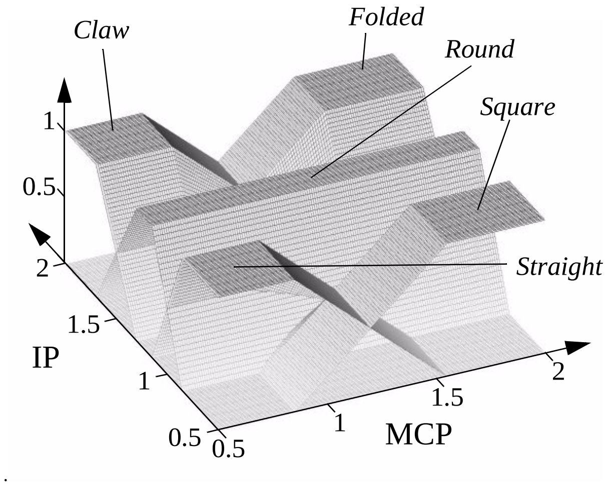

Fig. 5. Meanings of the items of $S_{flexion}$.

The illustration of this new distance is presented with the exemple of a finger flexion. Considering 2 numeric values of finger flexion $f_1 = (0.5, 0.5)$ and $f_2 = (1.2, 1.0)$ their fuzzy description are shown in fig. 6.

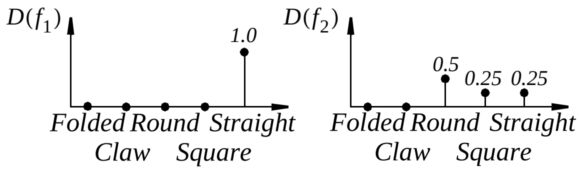

Fig. 6. Descriptions of f_1 and f_2.

The distance $d_{S_{flexion}}$ is arbitrary chosen as shown in Table 1. It it represents a human knowledge about the description of a finger flexion.

Table 1: Distance $d_{S_{flexion}}$ défined on $S_{flexion}$.

$d_{S_{flexion}}$	*Folded*	*Claw*	*Round*	*Square*	*Straight*
Folded	0	1	2	3	4
Claw	1	0	1	2	3
Round	2	1	0	1	2
Square	3	2	1	0	1
Straight	4	3	2	1	0

The distances to the two finger flexions f_1 and f_2 are calculated for any other finger posture (fig. 7 and 8). On both figs, five plates corresponding to the distance between each term and f_1 or f_2. The value on each plate is directly connected to the distance $d_{S_{flexion}}$.

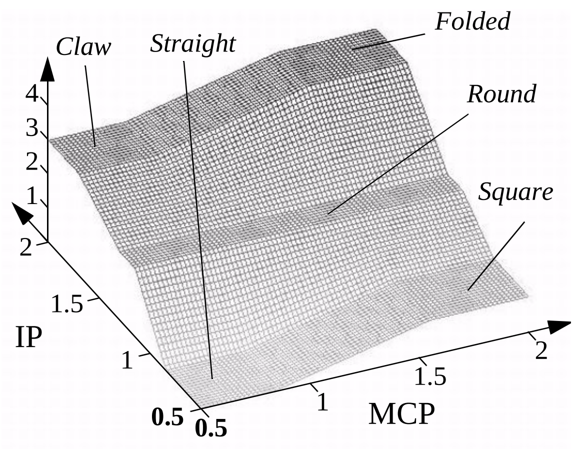

Fig. 7. $d_{tp}(D(f_1), D(g))$ **with** $f_1 = (0.5, 0.5)$ **.**

In Fig. Fig. 8. the values of the plates corresponding to the terms *Square* and *Round* are identical. This means that:

$$d(D(f_2), \{Square\}) = d(D(f_2), \{Round\}) \quad (23)$$

even if $\mu_{D(f_2)}(Round) \geq \mu_{D(f_2)}(Square)$ as shown in Fig. Fig. 6.. This result is consistent because the transportation distance takes all the terms into account. In this case f_2 is a little bit *Straight:* $\mu_{D(f_2)}(Straight) = 0.25$.

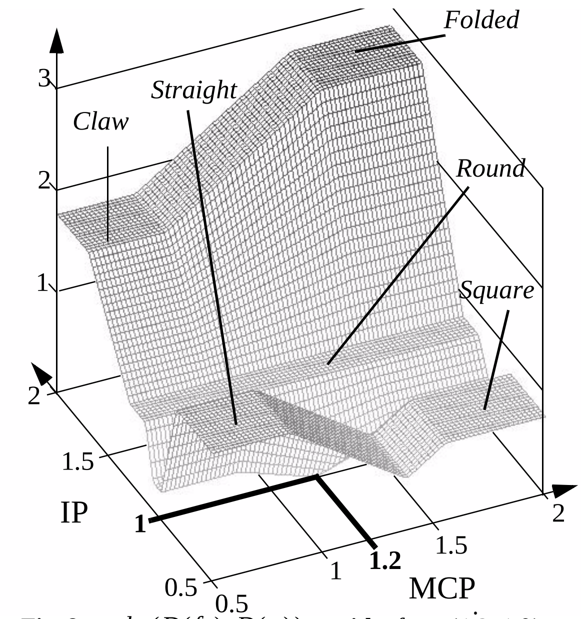

Fig. 8. $d_{tp}(D(f_2), D(g))$ **with** $f_2 = (1.2, 1.0)$ **.**

6. CONCLUSION

The formalization of the fuzzy subsets based description process of quantities is a slow process that needs to be performed in order to be able to manage this specific quantity representations. With the transportation distance, this paper gives a new tool for processing lexical fuzzy subsets issued from a measurement process. With its four properties: singleton coincidence, continuity, precision and constancy, the transportation is a good candidate to perform signal processing on this particular kind of representation issued from a fuzzy description of measurement. In this paper the distance betwen lexical term was issued from human knowledge, but it will be possible to extract it from a metric on physical states and from the fuzzy nominal scale. Then the scale will be enhanced in order to bring a metric from the set of physical states to the set of lexical fuzzy subsets.

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