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PRINCIPLE AND FUNDAMENTAL EQUATIONS OF GENERALIZED MEASUREMENT METHODS BASED ON LENGTH AND ANGULAR ACCUMULATION

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Abstract – A comprehensive and systematic description of measurement methods based on length and angular accumulation has not been reported yet. Based on deductions, the definition and fundamental equations of these methods are proposed and further generalized. Among the most significant is that the methods are understood and treated at a higher altitude and from a wider field of view. This is an extended abstract.

Keywords: generalization, length, angle, accumulation, measurement

1. INTRODUCTION

Measurement methods based on length and angular accumulation have been widely used in daily life along with areas including scientific research, industrial production, engineering surveying, and business activity for a long time. However, questions related to measurement methods involving length and angular accumulation are persistent. What can be measured? How these methods are characterized and classified? Is there a unified mathematical model? To the best of our knowledge, comprehensive and systematic answer for these questions is not available yet. Thus, in the manuscripts submitted recently by the author of this paper [1,2], a unified theory of the measurement methods is proposed on the basis of case analysis and induction. The theory primarily comprises the definition, classification, characteristics, and fundamental equations (unified mathematical model) of the measurement methods. However, because the theory is induction-based, it may include some errors or deficiencies because of the possible omission of some critical cases.

The initial objective of this paper is to derive the theory of the measurement methods on the basis of deductions; however, meanwhile, some instruments that are typically assumed to not be based on the measurement methods (e.g., dial indicators or lever indicators) were found to actually have opposite characteristics. Furthermore, it was noted that direct measurements, such as measuring length with a ruler, measuring arc length with a tape, or measuring angle with a protractor, can also be regarded as based on the measurement methods in a certain sense. Thus, in a broader sense, almost all mechanical instruments used to measure length and angle were found to possibly rely on the methods of interest in this paper. Therefore, compared with the initial objective, building a generalized uniform theory of the measurement methods based on deductions is more important.

This paper focuses on the principle and fundamental equations of the measurement methods.

2. MATHEMATICAL FOUNDATION

This section discusses the mathematical foundation of the generalized measurement methods based on length and angular accumulation. The relations among points and vectors in three-dimensional Euclidean space (R^3) are described.

2.1. Minimum displacement between two points (MDPs) and its synthesis

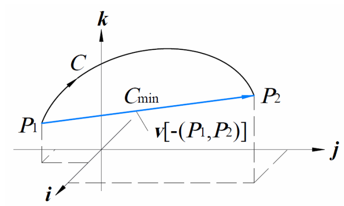

Fig.1. Minimum displacement between two points.

Definition 1: Vector of MDPs.
Given points P_1, $P_2 \in R^3$, let

$$\boldsymbol{v}[-(P_1,P_2)] = l[-(P_1,P_2)]\boldsymbol{\tau}[-(P_1,P_2)], \tag{1}$$

where $l[-(P_1,P_2)]$ and $\boldsymbol{\tau}[-(P_1,P_2)]$ represent the magnitude and unit direction vector of $-(P_1,P_2)$ (MDPs from P_1 to P_2), respectively. Then, $\boldsymbol{v}[-(P_1,P_2)]$ is defined as the vector of the MDPs from P_1 to P_2.

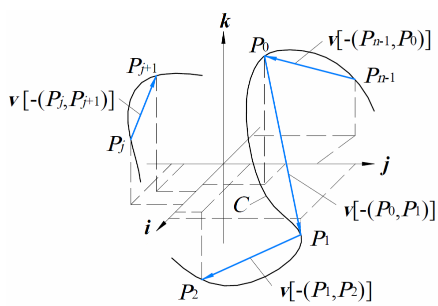

Fig.2. Closed synthesis of the MDPs from P_0, then successively through P_1, P_2, …, P_{n-1}, and finally to P_0.

As shown in Fig. 2, point P_0 moves along a closed curve C and back to its origin; meanwhile, it passes successively through points $P_1, P_2, \ldots, P_{n-1}$ $(2 \le n \le \infty)$, and $-(P_j,P_{j+1})$ $(j = 0, 1, \ldots, n-1)$ then form a closed synthesis of the MDPs from P_0 and then successively through $P_1, P_2, \ldots, P_{n-1}$, and finally to P_0. Even if C is not constant, this closed synthesis of MDPs is no longer related to C provided that it passes successively through P_j $(j = 0, 1, \ldots, n-1)$, and the following equation can be obtained:

$$\sum_{j=0}^{n-1} \boldsymbol{v}\left[-\left(P_j, P_{j+1}\right)\right] + P_0 = P_0 \,. \tag{2}$$

Note that (2) is a vector equation comprising at most three independent nonlinear equations; thus, using (2), at most three unknowns can be solved. This is the base of the generalized measurement method based on length accumulation defined below.

2.2. Minimum angular displacement between two vectors (MADVs) and its synthesis

As shown in Fig. 3, the initial points of unit vectors $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$ are both at the center of a unit sphere S. Let $\boldsymbol{L}_1$ make a fixed-point rotation or an angular displacement around the center of S such that $\boldsymbol{L}_1$ coincides with $\boldsymbol{L}_2$; meanwhile, the end point of $\boldsymbol{L}_1$ moves along a curve C on S. When C is the minor arc (C_{min}) of a great circle of S that connects the end points of $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$, the path of the end point of $\boldsymbol{L}_1$ is shortest, and the corresponding angular displacement of $\boldsymbol{L}_1$ is called the minimum angular displacement between two vectors (MADVs) from $\boldsymbol{L}_1$ to $\boldsymbol{L}_2$, denoted by $\angle(\boldsymbol{L}_1,\boldsymbol{L}_2)$.

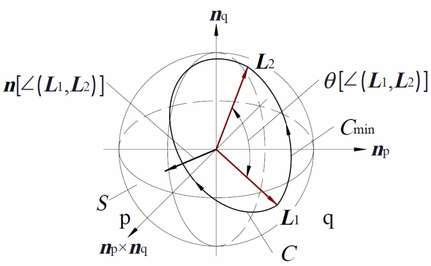

Fig.3. Minimum angular displacement between two vectors (I).

Obviously, $\angle(\boldsymbol{L}_1,\boldsymbol{L}_2)$ is a fixed-axis rotation of $\boldsymbol{L}_1$ wherein the direction of the axis that passes through the center of S is the same as $\boldsymbol{L}_1 \times \boldsymbol{L}_2$ and the angle of the rotation of $\boldsymbol{L}_1$ is equal to the arc length of $C_{\min}$, which are called the direction and magnitude of $\angle(\boldsymbol{L}_1,\boldsymbol{L}_2)$, respectively.

Definition 2: Tensor of MADVs[2].

Given nonzero vectors $\boldsymbol{L}_1, \boldsymbol{L}_2 \in \mathrm{R}^3$, let

$$\boldsymbol{T}[\angle(\boldsymbol{L}_1,\boldsymbol{L}_2)] = \cos\theta[\angle(\boldsymbol{L}_1,\boldsymbol{L}_2)]\boldsymbol{I} - \sin\theta[\angle(\boldsymbol{L}_1 \angle \boldsymbol{L}_2)]\boldsymbol{n}[\angle(\boldsymbol{L}_1,\boldsymbol{L}_2)]\cdot\boldsymbol{\varepsilon} \,, \tag{3}$$

where $\theta[\angle(\boldsymbol{L}_1,\boldsymbol{L}_2)]$ and $\boldsymbol{n}[\angle(\boldsymbol{L}_1,\boldsymbol{L}_2)]$ are the magnitude and unit direction vector of $\angle(\boldsymbol{L}_1,\boldsymbol{L}_2)$, respectively,

$$\boldsymbol{I} = \boldsymbol{ii} + \boldsymbol{jj} + \boldsymbol{kk} \text{ , and} \tag{4}$$

$$\boldsymbol{\varepsilon} = \boldsymbol{ijk} + \boldsymbol{jki} + \boldsymbol{kij} - \boldsymbol{kji} - \boldsymbol{jik} - \boldsymbol{ikj} \,, \tag{5}$$

where $\{\boldsymbol{i},\boldsymbol{j},\boldsymbol{k}\}$ is any orthonormal set of vectors in R^3. Then, $\boldsymbol{T}[\angle(\boldsymbol{L}_1,\boldsymbol{L}_2)]$ is defined as the tensor of the MADVs from $\boldsymbol{L}_1$ to $\boldsymbol{L}_2$.

As shown in Fig. 4, a nonzero vector $\boldsymbol{L}_0$ angularly displaces around its initial point so that its end point moves along a closed curve C and back to its origin; meanwhile, it coincides successively with nonzero vectors $\boldsymbol{L}_1, \boldsymbol{L}_2, \ldots, \boldsymbol{L}_{n-1}$ $(2 \le n \le \infty)$, whose initial points coincide with that of $\boldsymbol{L}_0$. Then, $\angle(\boldsymbol{L}_j, \boldsymbol{L}_{j+1})$ $(j = 0, 1, \ldots, n-1)$ form a closed synthesis of the MADVs from $\boldsymbol{L}_0$, then successively through $\boldsymbol{L}_1, \boldsymbol{L}_2, \ldots, \boldsymbol{L}_{n-1}$, and finally to $\boldsymbol{L}_0$. Even if C is not constant, this closed synthesis of MADVs is no longer related to C provided that it intersects successively those straight lines that coincides with $\boldsymbol{L}_j$ $(j=0,1,\ldots,n-1)$. Thus, the following equation can be obtained:

$$\prod_{j=n-1}^{0} \boldsymbol{T}\left[\angle\left(\boldsymbol{L}_j, \boldsymbol{L}_{j+1}\right)\right]\cdot \boldsymbol{L}_0 = \boldsymbol{L}_0 \,, \tag{6}$$

where the product in Π is the dot product.

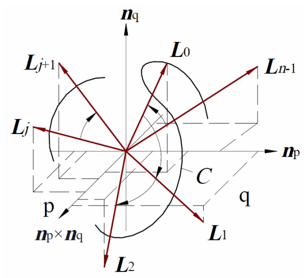

Fig.4. Closed synthesis of the MADVs from $\boldsymbol{L}_0$, then successively through $\boldsymbol{L}_1, \boldsymbol{L}_2, \ldots, \boldsymbol{L}_{n-1}$, and finally to $\boldsymbol{L}_0$.

Note that (6) is a tensor equation consisting of at most two independent linear equations; thus, it can solve at most two unknowns. This is the base of the generalized measurement method based on angular accumulation defined below.

2.3. Relation between MDPs and MADVs

The relation between MDPs and MADVs, which is the base of the three angular accumulation measurement methods defined below, is discussed in this section.

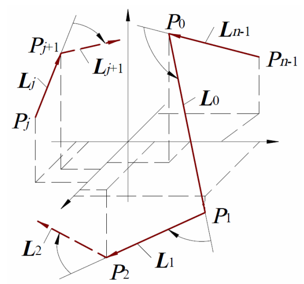

Fig.5. Relation between the closed syntheses of MDPs and MADVs: type of exterior angle.

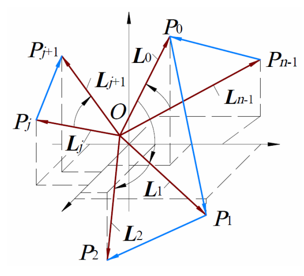

Fig.6. Relation between the closed syntheses of MDPs and MADVs: type of central angle.

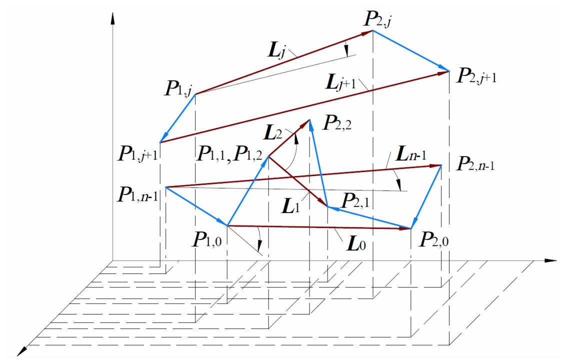

Fig.7. Relation between the closed syntheses of MDPs and MADVs: type of free angle.

3. BASIC CONCEPTS OF ACCUMULATION

This section defines some basic concepts of accumulation in measurement systems.

A postulate in this paper stipulates that the accumulation between two accumulative objects in a measuring system involves overlapping element between the accumulative objects.

Accumulation in a measuring system can be classified as either exterior or interior and occurs by means of contacting, aiming, or marking.

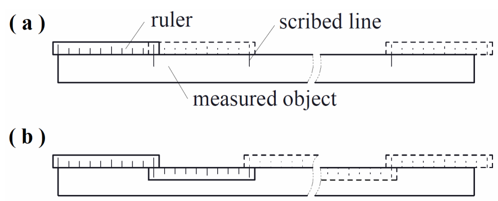

Fig.8. Ruler or accumulated rulers used to measure the length of an object: (a) interior accumulation and (b) exterior accumulation.

Definition 3: Accumulative point and central accumulative point (P).

Definition 4: Accumulative line of free angle between two accumulated objects ($\boldsymbol{L}$).

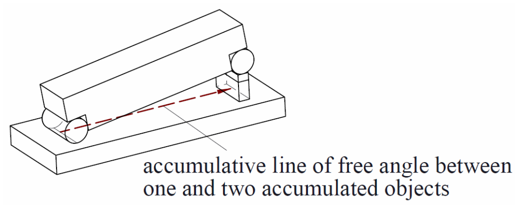

Fig.9. Accumulative line of free angle between one and two accumulated objects.

Definition 5: Accumulative line of free angle between one and two accumulated objects ($\boldsymbol{L}$).

Definition 6: Accumulative center (O) and accumulative line of central angle ($\boldsymbol{OP}$).

Definition 7: Accumulative line of exterior angle ($\boldsymbol{P}_j\boldsymbol{P}_{j+1}$).

4. GENERALIZED MEASUREMENT METHOD BASED ON LENGTH ACCUMULATION

In this section, the definition and fundamental equations of the generalized measurement method based on length accumulation are proposed, and some cases are verified.

Definition 8: Generalized measurement method based on length accumulation.

In a measuring system including the object to be measured, first, when it is in its initial configuration, build or select one or more groups of accumulative objects conforming to the following rule described in 1°, and select accumulative points or central accumulative points (if they exist) in each group conforming to the following rule described in 2° with a serial number assigned for each of them. Second, change the relative position of the accumulative objects in the measuring system to its current configuration. Then, if the measurand/s can be solved on the basis of the following equations (7) shown in 3°, such method used to obtain the measurand/s is called the generalized measurement method based on length accumulation corresponding to the group or groups selected and the accumulative points and central accumulative points selected in each group.

1° Every built or selected group of accumulative objects should be exclusive such that the accumulative objects are accumulated in series and closed.

2° In every built or selected group of accumulative objects, one accumulative point or central accumulative point (if one exists) should be selected between every two objects accumulated along the serial loop of the group, and they should be numbered in such a way that for every accumulative object, the numbers of the selected two accumulative points or central accumulative points are consecutive.

3° The equations based on which the measurand/s can be solved are as follows:

$$\begin{cases} \displaystyle\sum_{j=0}^{n_i-1} \boldsymbol{P}_{i,j}\boldsymbol{P}_{i,j+1}\left[\boldsymbol{l}(0)\right] = \boldsymbol{0} \\ \displaystyle\sum_{j=0}^{n_i-1} \left\{\boldsymbol{P}_{i,j}^{j}\boldsymbol{P}_{i,j}^{j+1}\left[\boldsymbol{l}(t)\right] + \boldsymbol{P}_{i,j}^{j+1}\boldsymbol{P}_{i,j+1}^{j+1}\left[\boldsymbol{l}(t)\right]\right\} = \boldsymbol{0} \end{cases},$$

$$i = 1,2,\ldots,m < \infty,\ 2 \le n_i \le \infty, \tag{7}$$

where t is the current time of the current configuration of the measuring system (t = 0 corresponds to the time of the initial configuration), $\boldsymbol{l}(0)$ is the measurand/s corresponding to the initial time t=0,$\boldsymbol{l}(t)$ is the measurand/s corresponding to the current time t, m is the total number of built or selected groups of accumulative objects, n_{i} is the total number of the accumulative objects in the ith group of accumulative objects, $P_{i,j}$ is the jth accumulative point or central accumulative points in the ith group of accumulative objects when the measuring system is in its initial configuration, $P_i^j{}_{,j}$ is the corresponding point of $P_{i,j}$ located at the jth accumulative object in the ith group of accumulative objects when the measuring system is in its current configuration, $P_i^{j+1}{}_{,j}$ is the corresponding point of $P_{i,j}$ located at the j+1th accumulative object in the ith group of accumulative objects when the measuring system is in its current configuration, where the jth accumulative object in the ith group of accumulative objects is the one at which the two accumulative points or central accumulative points located are j−1th and jth, respectively, $\boldsymbol{P}_{i,j}\boldsymbol{P}_{i,j+1}$ is the vector of the MDPs from $P_{i,j}$ to $P_{i,j+1}$, and $\boldsymbol{P}_i^{j+1}{}_{,j}\boldsymbol{P}_i^{j+1}{}_{,j+1}$ is the vector of the MDPs from $P_i^{j+1}{}_{,j}$ to $P_i^{j+1}{}_{,j+1}$.

Case 1: Measurement of the length of a circular arc subtended by two scribed lines with a tape.

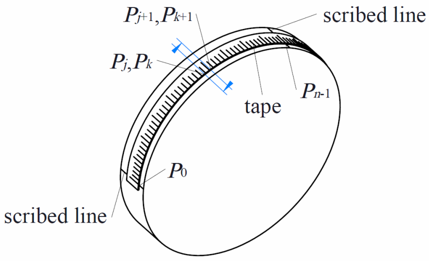

Fig.10. Measurement of the length of a circular arc subtended by two scribed lines with a tape.

By applying the first equation of (7) to the case 1, the following relation can be obtained:

$$\sum_{j=0}^{n-2} \boldsymbol{P}_j\boldsymbol{P}_{j+1} + \sum_{k=n-2}^{0} \boldsymbol{P}_{k+1}\boldsymbol{P}_k = \boldsymbol{0},$$

$$n \to \infty, \boldsymbol{P}_j\boldsymbol{P}_{j+1} \to \boldsymbol{0}, \boldsymbol{P}_{k+1}\boldsymbol{P}_k \to \boldsymbol{0}.$$

Note that

$$\boldsymbol{P}_{k+1}\boldsymbol{P}_k = -\boldsymbol{P}_j\boldsymbol{P}_{j+1}, \quad k = j, \quad j,k = 0,1,\ldots,\infty.$$

Then the measurand s is given by

$$s = \sum_{k=\infty}^{0} \left|\boldsymbol{P}_{k+1}\boldsymbol{P}_k\right| = \sum_{j=0}^{\infty} \left|\boldsymbol{P}_j\boldsymbol{P}_{j+1}\right| = s_1,$$

where s_1 is the reading of the tape.

Case 2: Measurement of the cone angle of an endocone with two balls.

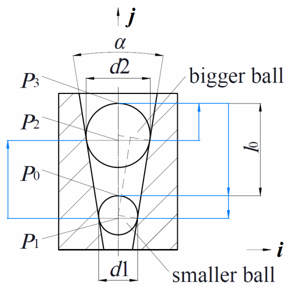

Fig.11. Measurement of the cone angle of an endocone with two balls.

By applying the first equation of (7) to the case 2, the following relation can be obtained:

$$\sum_{j=0}^{3} \boldsymbol{P}_j\boldsymbol{P}_{j+1} \cdot \boldsymbol{j} = 0.$$

Then the measurand α is given by

$$\alpha = 2\sin^{-1}\frac{d2 - d1}{2l_0 - d2 + d1},$$

where l_0 is what measured directly in the case.

Case 3: Measurement of the length of a part with a simplified dial indicator.

By applying (7) to the case 3 and subtracting the first equation of (7) from the second equation, the following relation can be obtained:

$$\boldsymbol{P}_1^1\boldsymbol{P}_1^2 + \boldsymbol{P}_1^2\boldsymbol{P}_2^2 + \boldsymbol{P}_2^2\boldsymbol{P}_2^3 + \boldsymbol{P}_3^3\boldsymbol{P}_3^0 - \boldsymbol{P}_1\boldsymbol{P}_2 = \boldsymbol{0}.$$

Hence, the measurand l is given by

$$\begin{aligned} l = \Delta l^3 &= -\boldsymbol{P}_3^3\boldsymbol{P}_3^0 \cdot \boldsymbol{j} \\ &= \boldsymbol{P}_1^1\boldsymbol{P}_1^2 \cdot \boldsymbol{j} + \boldsymbol{P}_1^2\boldsymbol{P}_2^2 \cdot \boldsymbol{j} + \boldsymbol{P}_2^2\boldsymbol{P}_2^3 \cdot \boldsymbol{j} - \boldsymbol{P}_1\boldsymbol{P}_2 \cdot \boldsymbol{j} \\ &= (r2\cos\varphi - r2) + (r1\sin\varphi - r2\cos\varphi) + (r1\varphi - r1\sin\varphi) - (-r2) \\ &= r1\varphi \\ &= \frac{r1}{r2}a \end{aligned}$$

where the relation of the length of $\boldsymbol{P}_2\boldsymbol{P}_2^3$ being equivalent to the length of the arc from P_2 to P_2^2 is used.

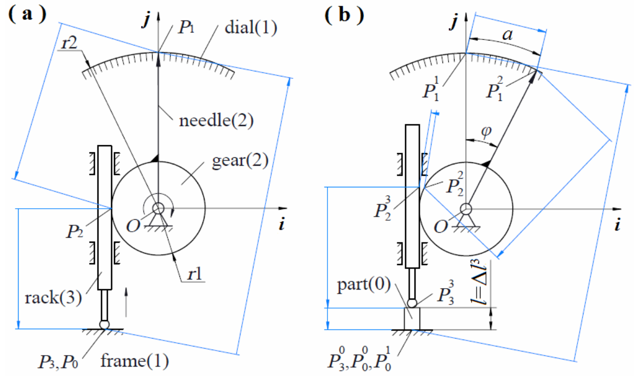

Fig.12. Measurement of the length of a part with a simplified dial indicator: (a) initial configuration and (b) current configuration.

Case 4: Measurement of the length of a part with a caliper.

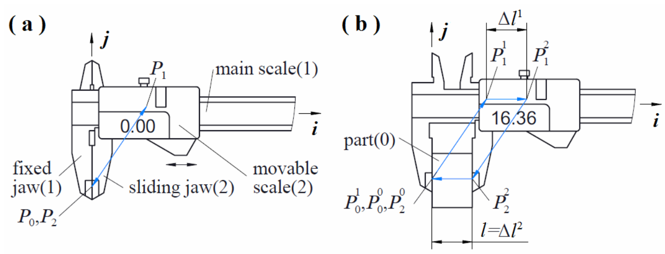

Fig.13. Measurement of the length of a part with a caliper: (a) initial configuration and (b) current configuration.

Similarly as in the case 3, the measurand l is given by

$$l = \Delta l^2 = \left|\boldsymbol{P}_2^2\boldsymbol{P}_2^0\right| = \left|\boldsymbol{P}_1^1\boldsymbol{P}_1^2\right| = \Delta l^1$$

where Δl^1 is the reading of the caliper.

Case 5: Measurement of the diameter of a cylinder with a flexible ring.

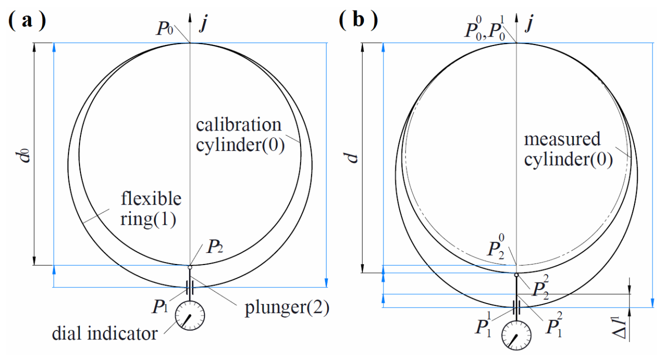

Fig.14. Measurement of the diameter of a cylinder with a flexible ring: (a) initial configuration and (b) current configuration.

Similarly as in the case 3, the measurand d is given by

$$\begin{aligned} d &= \boldsymbol{P}_2\boldsymbol{P}_0 \cdot \boldsymbol{j} - \left(\boldsymbol{P}_0^1\boldsymbol{P}_1^1 - \boldsymbol{P}_0\boldsymbol{P}_1\right)\cdot \boldsymbol{j} - \boldsymbol{P}_1^1\boldsymbol{P}_1^2 \cdot \boldsymbol{j} \\ &= d_0 + k\Delta F - \Delta l^1 \end{aligned}.$$

where k is the stiffness of the ring, ΔF is the increment of the measuring force, Δl^1 is the reading of the dial indicator.

Case 6: Measurement of the run-out of a circle.

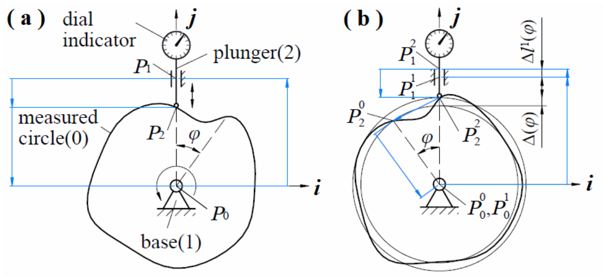

Fig.15. Measurement of the run-out of a circle: (a) initial configuration and (b) current configuration.

Similarly as in the case 3, the following relation can be obtained:

$$\Delta(\varphi) = \left[-\left(\boldsymbol{P}_2^2\boldsymbol{P}_2^0 + \boldsymbol{P}_2^0\boldsymbol{P}_0^0\right) + \boldsymbol{P}_2\boldsymbol{P}_0\right]\cdot \boldsymbol{j} = \boldsymbol{P}_1^1\boldsymbol{P}_1^2 \cdot \boldsymbol{j} = \Delta l^1(\varphi)$$

where $\Delta l^1(\varphi)$ is the reading of the dial indicator. Hence, the measurand Δ is given by

$$\Delta = \max[\Delta(\varphi)] - \min[\Delta(\varphi)], \quad \varphi \in [0, 2\pi).$$

5. GENERALIZED MEASUREMENT METHOD BASED ON ANGULAR ACCUMULATION

In this section, the definitions and fundamental equations of the three types of generalized measurement methods based on angular accumulation are proposed, and some cases are verified.

Definition 9: Generalized measurement method based on the accumulation of exterior angles.

$$\prod_{j=n-1}^{0} \boldsymbol{T}\left[\angle\left(\boldsymbol{P}_j\boldsymbol{P}_{j+1}, \boldsymbol{P}_{j+1}\boldsymbol{P}_{j+2}\right)(\theta)\right]\cdot \boldsymbol{P}_0\boldsymbol{P}_1 = \boldsymbol{P}_0\boldsymbol{P}_1 . \tag{8}$$

Case 7: Measurement of an interior angle of a triangle.

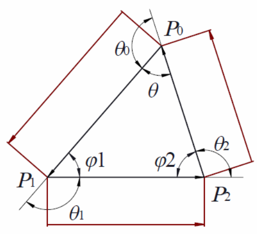

Fig.16. Measurement of an interior angle of a triangle by means of measuring the other two interior angles of the triangle.

By applying (8) to the case 7, the following relation can be obtained:

$$\sum_{j=0}^{2} \theta_j = 2\pi .$$

Then the measurand θ is given by

$$\begin{aligned} \theta &= \pi - \theta_0 = \pi - \left[2\pi - \theta_1 - \theta_2\right] \\ &= \pi - \varphi 1 - \varphi 2 \end{aligned},$$

where $\varphi 1$ and $\varphi 2$ are the two other interior angles of the triangle and measured directly in the case.

Definition 10: Generalized measurement method based on the accumulation of central angles.

$$\begin{cases} \prod_{j=n-1}^{0} \boldsymbol{T}\left\{\angle\left(\boldsymbol{OP}_j, \boldsymbol{OP}_{j+1}\right)[\theta(0)]\right\}\cdot \boldsymbol{OP}_0 = \boldsymbol{OP}_0 \\ \prod_{j=n-1}^{0} \left\{\boldsymbol{T}\left\{\angle\left(\boldsymbol{OP}_j^{j+1}, \boldsymbol{OP}_{j+1}^{j+1}\right)[\theta(t)]\right\}\right. \\ \quad \left.\cdot \boldsymbol{T}\left\{\angle\left(\boldsymbol{OP}_j^{j}, \boldsymbol{OP}_j^{j+1}\right)[\theta(t)]\right\}\right\}\cdot \boldsymbol{OP}_0^0 = \boldsymbol{OP}_0^0 \end{cases} . \tag{9}$$

Case 8: Measurement of the ruling spans of a dial.

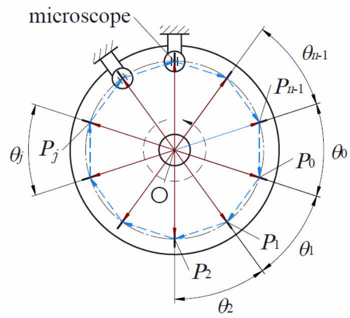

Fig.17. Measurement of the ruling spans of a dial based on the relation that the sum of all ruling spans of the dial is 2π.

By applying the first equation of (9) to the case 8, the following relation can be obtained:

$$\sum_{j=0}^{n-1} \theta_j = \sum_{j=0}^{n-1} \left(\theta_j^{'} + \Delta\right) = 2\pi ,$$

where θ_j and $\theta_j^{'}$ (j=0,1,…,n−1) are the ruling spans of the dial and corresponding readings of the two microscopes respectively, Δ is the systematic error of the readings of the two microscopes and obviously,

$$\Delta = \frac{2\pi - \sum_{j=0}^{n-1} \theta_j^{'}}{n} .$$

Case 9: Measurement of the angle between two bolt-holes of a bolt-sphere.

By applying (9) to the case 9, the following relation can be obtained:

$$\boldsymbol{OP}_3 = \boldsymbol{T}\left[\angle\left(\boldsymbol{OP}_3^3, \boldsymbol{OP}_3^0\right)\right]\cdot \boldsymbol{T}[(1),(0)]\cdot \boldsymbol{T}[(2),(1)]\cdot \boldsymbol{OP}_3 ,$$

where $\boldsymbol{T}$[(1),(0)] and $\boldsymbol{T}$[(2),(1)] are the tensors of the rigid body rotation of the cantilever crane(1) relative to the base(0) and the slider(2) to the cantilever crane(1) respectively[3].

Then, the measurand θ is given by

$$\begin{aligned} 0 \le \theta &= \Delta\theta^3 = \theta\left[\angle\left(\boldsymbol{OP}_3^3, \boldsymbol{OP}_3^0\right)\right] \\ &= \cos^{-1}\left\{\frac{\{\boldsymbol{T}[(1),(0)]\cdot \boldsymbol{T}[(2),(1)]\cdot \boldsymbol{OP}_3\}\cdot \boldsymbol{OP}_3}{\left|\boldsymbol{T}[(1),(0)]\cdot \boldsymbol{T}[(2),(1)]\cdot \boldsymbol{OP}_3\right|\left|\boldsymbol{OP}_3\right|}\right\} \le \pi \end{aligned} .$$

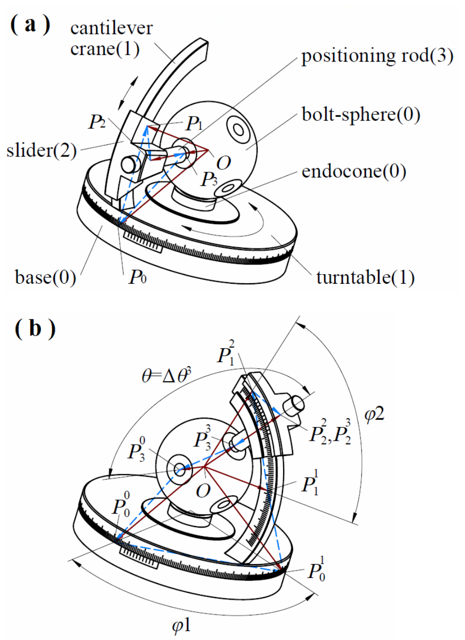

Fig.18. Measurement of the angle between two bolt-holes of a bolt-sphere: (a) initial configuration and (b) current configuration.

Definition 11: Generalized measurement method based on the accumulation of free angles.

$$\begin{cases}\prod_{j=n_i-1}^{0} \boldsymbol{T}\{\angle(\boldsymbol{L}_{i,j},\boldsymbol{L}_{i,j+1})[\theta(0)]\}\cdot \boldsymbol{L}_{i,0}=\boldsymbol{L}_{i,0}\\ \prod_{j=n_i-1}^{0}\{\boldsymbol{T}\{\angle(\boldsymbol{L}_{i,j}^{j+1},\boldsymbol{L}_{i,j+1}^{j+1})[\theta(t)]\} \\ \quad\cdot \boldsymbol{T}\{\angle(\boldsymbol{L}_{i,j}^{j},\boldsymbol{L}_{i,j}^{j+1})[\theta(t)]\}\}\cdot \boldsymbol{L}_{i,0}^{0}=\boldsymbol{L}_{i,0}^{0}\end{cases} . \qquad (10)$$

Case 10: Measurement of space angles with a simplified articulated CMM.

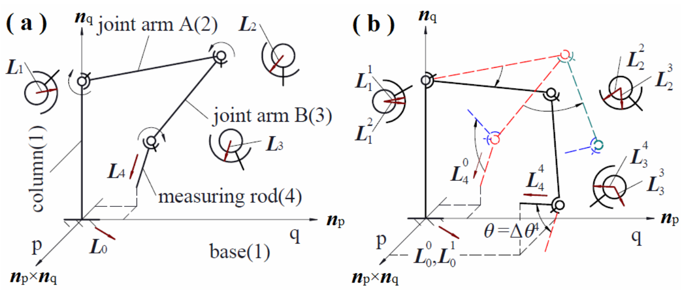

Fig.19. Measurement of space angles with an articulated CMM: (a) initial configuration and (b) current configuration.

By applying (10) to the case 10, the following relation can be obtained:

$$\boldsymbol{L}_4=\boldsymbol{T}[\angle(\boldsymbol{L}_4^4,\boldsymbol{L}_4^0)]\cdot\prod_{j=1}^{3}\boldsymbol{T}[(k+1),(k)]\cdot\boldsymbol{L}_4,$$

where $\boldsymbol{T}[(2),(1)]$, $\boldsymbol{T}[(3),(2)]$ and $\boldsymbol{T}[(4),(3)]$ are the tensors of the rigid body rotation of the joint arm A(2) relative to the column(1), the joint arm B(3) to the joint arm A(2) and the measuring rod(4) to the joint arm B(3), respectively[3].

Then, the measurand θ can be obtained in the same way as in the case 9.

6. CONCLUSIONS

Generalized measurement methods based on length and angular accumulation are widely applied; thus, the establishment of a unified theory of such methods is important.

On the basis of deductions, this study proposes the definition and fundamental equations of these methods.

The principle of the generalized measurement methods based on length and angular accumulation is as follows. First, several unknown quantities can be mathematically determined from a closed synthesis of MDPs or MADVs in three-dimensional Euclidean space. Second, based on the accumulation of accumulative objects in a measuring system due to contacting, aiming, or marking, the closed synthesis of MDPs or MADVs can be obtained physically in real three-dimensional space by selecting groups of accumulative objects and accumulative points or central accumulative points in the groups; thus, the measurand related to them can be solved.

Two points were noted. First, for exterior accumulation, the accumulation made by contacting and aiming between the accumulative objects corresponds roughly to the positioning and aiming, respectively, usually found in the description of the instrument used for measurements of length or angle. This means that the measurement methods essentially describe the principle of the instrument from the viewpoint of accumulation. Second, for interior accumulation, the marks that divide virtually the object to be measured into accumulative objects are the surface elements of the object that exist naturally or are artificially imposed. This means that the methods essentially describe the method used to determine the measurand that maps the relations among the surface elements of the measured object, also from the viewpoint of accumulation.

Therefore, what described in this paper is not a new principle but a new description. Among the most significant of this description is that it leads the measurement methods based on length and angular accumulation to be understood and treated at a higher altitude and from a wider field of view.

Note that (7)-(10) are depending only on the initial and current configurations of the measuring system and unrelated with the process of transition. It means that the boundaries of validity of the methods may be very extensive, which is needed to be further studied.

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