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COMPARISON OF SOME ALGORITHMS FOR THREE MEASUREMENTS EVALUATION

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Abstract - This work deals with evaluation of the measurand when results of only three measurements are available. Different algorithms are analyzed and compared according to the methodology of data processing algorithms certification.

Keywords: measurement data, expectation, probability distribution, variance, coverage interval

1. INTRODUCTION

Data evaluation algorithms applied in metrology are based on the methods of probability theory and mathematical statistics. But methodology for investigation of these algorithms is quite different from that is realized in statistics. The statistical criteria are based on assumptions about the form of probability distribution function, and the corresponding estimators performance is usually studied at a large number of repeated data. In metrology the real situation differs from the above. Quite often there are only few measurement data and no sound arguments for specifying the probability distribution function.

The paper deals with the task of measurand evaluation when only three measurement results are available: x_1, x_2, x_3. This situation is quite usual in metrological practice. Most of measurements are carried out according to certified measurement procedures, for which measurement accuracy is prior estimated and expressed, e.g., as a maximum permissible error or target measurement uncertainty. In these cases repeated measurements are carried out for control purposes only (not for measurement uncertainty calculations). Two or three measurements allow to avoid outliers in measurement and to check that measurement process is under the control.

A lot of heuristic algorithms can be suggested for evaluation the result of three measurements [1]. Here only few of them are taken for consideration, namely the following algorithms (estimators) are analyzed and compared:

$$
\begin{aligned}
m_1(x_1,x_2,x_3) &= \bar{x} \equiv (x_1+x_2+x_3)/3,\\
m_2(x_1,x_2,x_3) &= \sqrt{(x_1^2+x_2^2+x_3^2)/3},\\
m_3(x_1,x_2,x_3) &= \sqrt[3]{x_1\,x_2\,x_3},\\
m_4(x_1,x_2,x_3) &= x_{(2)}, \quad x_{(1)} \le x_{(2)} \le x_{(3)},\\
m_5(x_1,x_2,x_3) &= \left(x_{(1)}+x_{(3)}\right)/2;
\end{aligned}
$$

here m_1 is a simple mean, m_2 is RMS, m_3 is a geometric mean, m_4 is a median of an order statistics $\{x_{(i)}\}$, m_5 is a minimax estimate. The estimator m_2 is considered for distributions with non-negative domain.

In [2] simple mean, median and minimax estimates were compared according to their variances for several probability distributions (normal, uniform, triangle and arcsine distributions).

In this paper the algorithms are compared according to the methodology of data processing algorithms certification [3]. The idea of algorithm certification is to investigate the performance of a group of the algorithms used for solving the same task. For each algorithm several performance characteristics are calculated; the results are compared with the same characteristics obtained for the other algorithms from the group. Below the following characteristics are considered: bias, variance and coverage (probability) interval.

2. MODELING AND DATA EVALUATION

Measurement data x_1, x_2, x_3 are supposed to be realizations of independent identically distributed random quantities $X_{1,2,3} \in F(m)$, m is a parameter of the distribution function supposed to be a measurand value (reference value). Usually m equals to the expectation but for asymmetrical distributions we additionally consider the case when m is a distribution mode. The following probability distribution are considered: normal, uniform, exponential, triangular, gamma, arcsine etc.

Using Monte Carlo simulation method the distributions of the above estimates were obtained. Some of these distributions are presented below. Performance characteristics (bias, variance and probability interval for confidence level 0.95) are calculated. The bias is defined as a deviation of the estimator expectation Em_i from the reference value m: $b_i = Em_i - m$.

The estimators considered are compared with each other and with the maximum likelihood estimates (MLE).

2.1. Normal Distribution

Consider $F(m) = \mathcal{N}(m, 1/3)$. Let us use the parameter value $m = 0$ for Monte Carlo simulation (hereafter the number of simulated 3-element samples is $5 \cdot 10^6$). Maximum likelihood estimator here coincides with the simple mean: $\hat{m} \equiv m_1(x_1,x_2,x_3)$. The performance characteristics are given in the Table 1 (here

I is a notation for the coverage interval; confidence level is 0.95):

Table 1. Simulation results; $X_i \in \mathcal{N}(0, 1/3)$.

	Em_i	b_i	$\mathrm{Var}\, m_i$	I_{m_i}
m_1	0	0	0.037	[-0.38, 0.38]
m_3	0	0	0.057	-
m_4	0	0	0.050	[-0.44, 0.44]
m_5	0	0	0.040	[-0.39, 0.39]

The pdfs for the estimators are presented on the Fig. 1.

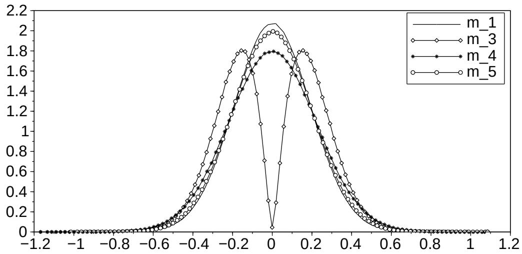

Fig. 1. pdf plots for m_j; $X_i \in \mathcal{N}(0, 1/3)$

Note that estimators m_1 and m_5 demonstrate close characteristics. Estimator m_3 is not applicable in that case.

2.2. Uniform Distribution

Let $F(m) = \mathrm{U}(0, 2m)$. Maximum likelihood estimator (unbiased) is $\hat{m} = \frac{1}{2}\,\frac{4}{3}\,x_{(3)} = \frac{2}{3}\,x_{(3)}$. Let us use $m = \frac{1}{2}$ for the simulation. The performance characteristics and pdfs are given in Table 2 and Fig. 2, correspondingly.

Table 2. Simulation results; $X_i \in \mathrm{U}(0, 1)$.

	Em_i	b_i	$\mathrm{Var}\, m_i$	I_{m_i}
m_1	0.500	0.000	0.028	[0.18, 0.82]
m_2	0.554	0.054	0.026	[0.22, 0.84]
m_3	0.422	-0.078	0.038	[0.07, 0.80]
m_4	0.500	0.000	0.050	[0.09, 0.91]
m_5	0.500	0.000	0.025	[0.18, 0.82]
$\hat{m}$	0.500	0.000	0.017	[0.25, 0.67]

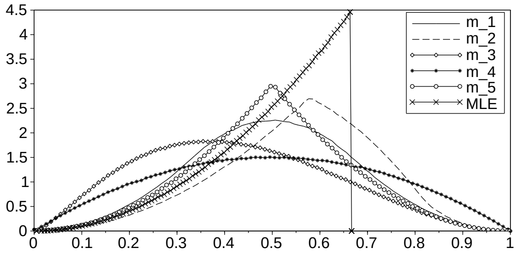

Fig. 2. pdf plots for m_j; $X_i \in \mathrm{U}(0, 1)$

The estimators m_1 and m_5 demonstrate close characteristics, but the MLE's ones are significantly better. It should be pointed that the estimators m_1, m_4, m_5 and MLE are unbiased.

2.3. Exponential Distribution

Let $F(m) = \mathrm{Exp}(1/m)$. Maximum likelihood estimator is $\hat{m} \equiv m_1(x_1, x_2, x_3)$. Let us use $m = 2$ for the simulation; the resulting data are presented below (see Table 3, Fig. 3). Note that only $\hat{m} \equiv m_1$ provides an unbiased estimate.

Table 3. Simulation results; $X_i \in \mathrm{Exp}(1/2)$.

	Em_i	b_i	$\mathrm{Var}\, m_i$	I_{m_i}
m_1	2.000	0.000	1.333	[0.21, 4.26]
m_2	2.429	0.431	2.090	[0.23, 5.25]
m_3	1.423	-0.576	0.914	[0.06, 3.29]
m_4	1.666	-0.333	1.444	[0.03, 4.01]
m_5	2.165	0.167	1.694	[0.18, 4.69]

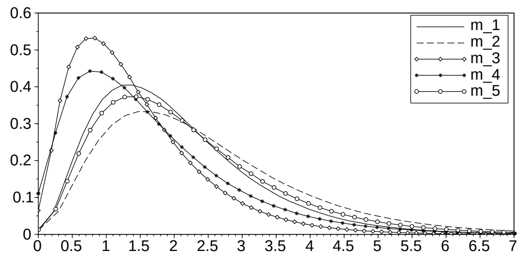

Fig. 3. pdf plots for m_j; $X_i \in \mathrm{Exp}(1/2)$

2.4. Triangular Distribution

Let $F(m)$ be a triangular distribution $F(m) = \mathrm{Tr}(-1, m, 1)$ with support $[-1, 1]$ and mode m, which is symmetrical in case $m = 0$ and asymmetrical otherwise. Maximum likelihood estimator $\hat{m}$ for m is [4]

$$\hat{m} = x_{(i_0)}, \quad i_0 = \arg \max_{1 \le i \le 3} M(i),$$

where $\{x_{(i)}\}$ is the order statistics, and

$$M(i) = \prod_{j=1}^{i-1} \frac{x_{(j)} + 1}{x_{(i)} + 1} \prod_{j=i+1}^{3} \frac{1 - x_{(j)}}{1 - x_{(i)}}.$$

The simulation results for the symmetrical case ($m = 0$) are given in Table 4 and Fig. 4.

Note that m_1 and m_5 demonstrate very close (and the most reasonable) behavior for that case. The MLE has the largest variance, and the corresponding pdf is multimodal. All the estimates are unbiased here.

But in case of the asymmetrical distribution all the estimates are biased; and $\hat{m}$ shows the minimal bias (and its mode is very close to m). See, e.g., simulation results for $m = 0.7$ given in Table 5 and Fig. 5. Note also that in case if suppose $m = EX$, then the estimate m_1 is unbiased.

Table 4. Simulation results; $X_i \in$ Tr$(-1, 0, 1)$.

	Em_i	b_i	Var m_i	I_{m_i}
m_1	0	0	0.056	[-0.46, 0.46]
m_3	0	0	0.091	-
m_4	0	0	0.082	[-0.57, 0.57]
m_5	0	0	0.057	[-0.46, 0.46]
$\hat{m}$	0	0	0.272	-

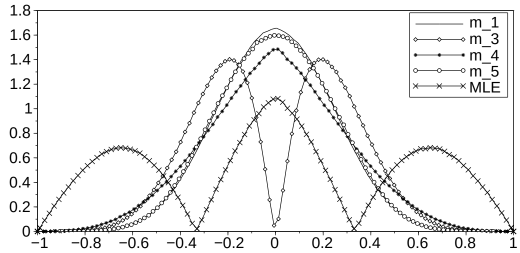

Fig. 4. pdf plots for m_j; $X_i \in$ Tr$(-1, 0, 1)$

Table 5. Simulation results; $X_i \in$ Tr$(-1, 0.7, 1)$.

	Em_i	b_i	Var m_i	I_{m_i}
m_1	0.233	-0.467	0.065	[-0.25, 0.71]
m_3	0.037	-0.663	0.157	-
m_4	0.265	-0.435	0.102	[-0.37, 0.80]
m_5	0.217	-0.483	0.064	[-0.26, 0.70]
$\hat{m}$	0.415	-0.285	0.244	-

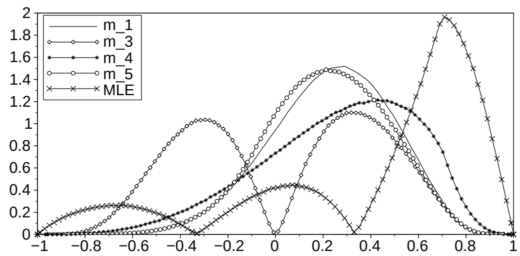

Fig. 5. pdf plots for m_j; $X_i \in$ Tr$(-1, 0.7, 1)$

2.5. Beta Distribution

Let $X_i \in$ Beta(α, β). Consider the following cases:

1. $\alpha = m$, $\beta = 1 - m$, where $m < 1$. In such a case $m = EX$. Let, e.g., $m = 0.5$ (arcsine distribution). The performance characteristics and pdf distributions are given in Table 6 and Fig. 6, correspondingly. Here m_5 demonstrates the best behavior from the estimators considered.

Table 6. Simulation results; $X_i \in$ Beta$(0.5, 0.5)$.

	Em_i	b_i	Var m_i	I_{m_i}
m_1	0.500	0.000	0.042	[0.10, 0.90]
m_2	0.580	0.080	0.038	[0.15, 0.92]
m_3	0.363	-0.137	0.063	[0.00, 0.83]
m_4	0.500	0.000	0.087	[0.02, 0.98]
m_5	0.500	0.000	0.034	[0.11, 0.89]

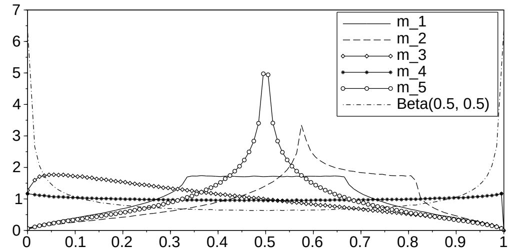

Fig. 6. pdf plots for m_j; $X_i \in$ Beta$(0.5, 0.5)$

2. $\alpha = \beta > 1$. Then $m \equiv \frac{1}{2}$. Let, e.g., $\alpha = \beta = 2$ (see Table 7, Fig. 7):

Table 7. Simulation results; $X_i \in$ Beta$(2, 2)$.

	Em_i	b_i	Var m_i	I_{m_i}
m_1	0.500	0.000	0.017	[0.25, 0.75]
m_2	0.533	0.033	0.016	[0.28, 0.77]
m_3	0.459	-0.041	0.020	[0.19, 0.73]
m_4	0.500	0.000	0.027	[0.19, 0.81]
m_5	0.500	0.000	0.016	[0.25, 0.75]

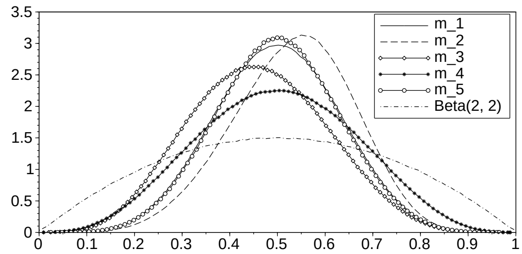

Fig. 7. pdf plots for m_j; $X_i \in$ Beta$(2, 2)$

Here m_1 and m_5 demonstrate the similar performance characteristics (better than the other estimators' characteristics).

3. $\alpha > \beta > 1$. Let us obtain maximum likelihood estimators for α and β according to [5], as a numerical solution for the system:

$$\psi(\hat{\alpha}) - \psi(\hat{\alpha} + \hat{\beta}) = \ln\left(\prod_1^3 x_i^{(1/3)}\right),$$

$$\psi(\hat{\beta}) - \psi(\hat{\alpha} + \hat{\beta}) = \ln\left(\prod_1^3 (1 - x_i)^{(1/3)}\right),$$

where ψ is the digamma function.

For Beta(α, β) suppose $m = \frac{\alpha-1}{\alpha+\beta-2}$ (its mode), so in addition to m_1 - m_5 let us consider also an estimator $m_6 = \frac{\hat{\alpha}-1}{\hat{\alpha}+\hat{\beta}-2}$, for α and β.

Below the simulation results for $\alpha = 5$ and $\beta = 2$ are given (see Table 8, Fig. 8); for m_6 we used 10^5 3-element samples. Suppose mode ($m = 0.8$) to be the reference value.

Table 8. Simulation results; $X_i \in$ Beta$(5, 2)$.

	Em_i	b_i	Var m_i	I_{m_i}
m_1	0.714	-0.086	0.009	[0.53, 0.88]
m_2	0.727	-0.073	0.008	[0.55, 0.89]
m_3	0.701	-0.099	0.010	[0.50, 0.89]
m_4	0.725	-0.075	0.012	[0.50, 0.92]
m_5	0.708	-0.092	0.009	[0.52, 0.88]
m_6	0.775	-0.025	0.453	[0.54, 1.04]

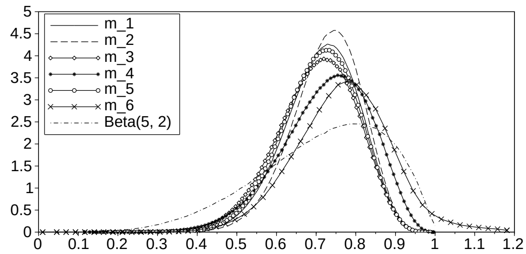

Fig. 8. pdf plots for m_j; $X_i \in$ Beta$(5, 2)$

The bias b_6 has the minimal value from considered ones (but Var m_6 is maximal).

If suppose $m = EX$ in the latter example then the bias for m_1 would be zero. Note also that distributions for m_1 and $\frac{\hat{\alpha}}{\hat{\beta}}$ are coincide (the same holds true for the cases 1, 2).

We did not consider the cases when $\alpha = 1$ or $\beta = 1$.

2.6. Gamma Distribution

Suppose that $X_i \in$ Gamma(α, β). Consider the following cases:

1. $\alpha \leq 1$. In such a case $m = EX$ and the mode is equal to 0. The simulation results for Gamma$(1, 0.5)$ are presented below (see Table 9, Fig. 9).

2. $\alpha > 1$. For Gamma(α, β) suppose $m = \frac{\alpha-1}{\beta}$ (its mode), so in addition to m_1 - m_5 let us consider also an estimator $m_6 = \frac{\hat{\alpha}-1}{\hat{\beta}}$, where $\hat{\alpha}$ and $\hat{\beta}$ are maximum likelihood estimators for α and β, obtained according to the procedure provided in [6]. Simulation results for Gamma$(5, 1)$ are presented in the Table 10, see also Fig. 10 (mode is equal to 4 here).

Table 9. Simulation results; $X_i \in$ Gamma$(1, 0.5)$.

	Em_i	b_i	Var m_i	I_{m_i}
m_1	2.000	0.000	1.332	[0.16, 4.21]
m_2	2.430	0.430	2.088	[0.23, 5.26]
m_3	1.424	-0.576	0.914	[0.04, 3.27]
m_4	1.667	-0.333	1.443	[0.05, 4.01]
m_5	2.166	0.166	1.692	[0.20, 4.69]

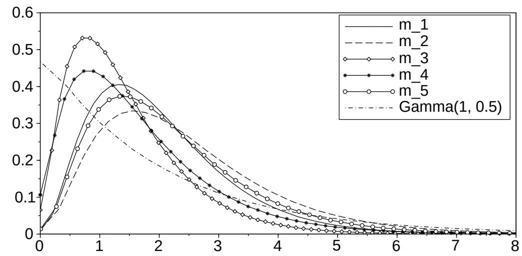

Fig. 9. pdf plots for m_j; $X_i \in$ Gamma$(1, 0.5)$

Table 10. Simulation results; $X_i \in$ Gamma$(5, 1)$.

	Em_i	b_i	Var m_i	I_{m_i}
m_1	5.000	1.000	1.667	[2.60, 7.53]
m_2	5.298	1.298	1.945	[2.67, 8.00]
m_3	4.676	0.676	1.558	[2.30, 7.07]
m_4	4.820	0.820	2.152	[2.06, 7.64]
m_5	5.092	1.092	1.877	[2.55, 7.76]
m_6	4.334	0.334	1.697	[1.86, 6.90]

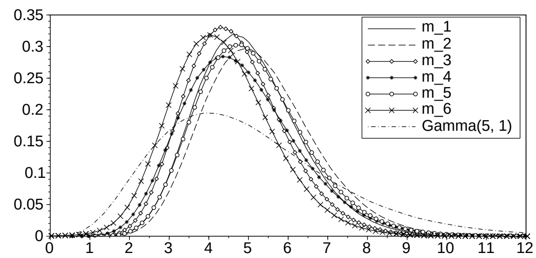

Fig. 10. pdf plots for m_j; $X_i \in$ Gamma$(5, 1)$

If suppose $m = EX$ in the latter example then the bias for m_1 would be zero.

2.7. Cauchy Distribution

Consider the Cauchy distribution with the location parameter m and fixed shape parameter equal to 1. This distribution type is very specific as it does not have finite moments of order greater than or equal to one. Besides m_i let us also calculate the maximum likelihood estimator for m using the following formula [7]:

$$\hat{m} = \big((x_{(3)} - x_{(2)})^2 + x_{(2)}(x_{(3)} - x_{(1)})^2 + \\ + x_{(3)}(x_{(2)} - x_{(1)})^2\big) \times \\ \times \big((x_{(3)} - x_{(2)})^2 + (x_{(3)} - x_{(1)})^2 + (x_{(2)} - x_{(1)})^2\big)^{-1}.$$

Simulation results for $m = 0$ and 4 are presented below (Table 11, Fig. 11 for $m = 0$; Table 12, Fig. 12 for $m = 4$). Only m_3, m_4 and $\hat{m}$ are considered; other estimates have distributions with too heavy tails. Note also that m_4 and $\hat{m}$ provide very close results (and the responding estimates are unbiased).

Table 11. Simulation results; $X_i \in$ Cauchy$(0, 1)$.

	Em_i	b_i	Var m_i	I_{m_i}
m_3	0	0	7.64	[-4.40, 4.40]
m_4	0	0	9.11	[-3.27, 3.27]
$\hat{m}$	0	0	8.51	[-3.20, 3.20]

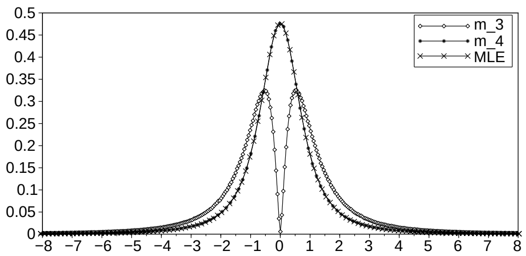

Fig. 11. pdf plots for m_j; $X_i \in$ Cauchy$(0, 1)$

Table 12. Simulation results; $X_i \in$ Cauchy$(4, 1)$.

	Em_i	b_i	Var m_i	I_{m_i}
m_3	2.58	-1.42	27.55	[-8.47, 9.57]
m_4	4.00	0.00	9.11	[0.76, 7.27]
$\hat{m}$	4.00	0.00	8.51	[0.80, 7.20]

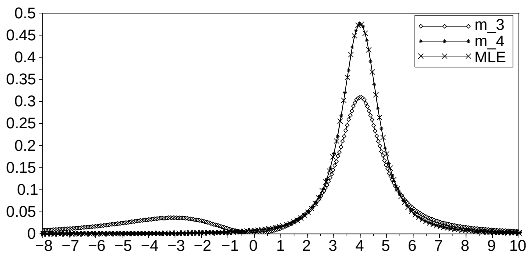

Fig. 12. pdf plots for m_j; $X_i \in$ Cauchy$(4, 1)$

3. CONCLUSIONS

The paper discusses the algorithms used for evaluation of result of three measurements. Different heuristic algorithms are analyzed using the methodology of data processing algorithms certification. The performance characteristics of the algorithms are obtained using Monte Carlo simulation. The algorithms are compared with each other and with the maximum likelihood estimates.

The simple mean and minimax estimates demonstrate similar performance characteristics for many cases considered. As a rule the simple mean provides unbiased estimates if a reference value is chosen as an expectation of the original distribution. For asymmetrical original distributions the MLE mode is often numerically coincides with the mode of the original distribution.

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