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MEASUREMENT UNCERTAINTY AND FUZZY MODELS

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Abstract - Measurement results of continuous quantities are not precise numbers, but more or less non-precise. They are connected with different kinds of uncertainty: Variability, errors, and fuzziness. This fuzziness of individual measurement results is different from errors, and the best up to date description of this imprecision is by fuzzy numbers. Examples will be given how to obtain the characterizing functions of fuzzy data, and to illustrate the generalized analysis method, which uses fuzzy models.

Keywords: characterizing function, fuzzy mean value, fuzzy numbers, fuzzy standard deviation, measurement result

1. INTRODUCTION

All the measurements of continuous quantities are containing different kinds of uncertainty, as is mentioned in [1] and also in related references as [2, 3], but usually mainly statistical variability and errors are considered. Considering a measurement result of an one-dimensional physical quantity, the idealized result is a real number times the measurement unit, but it is impossible to obtain all the infinitely many digits. Therefore the measurement result cannot be a real number in the mathematical strict sense, but in some way a generalized real number, which can be represented by so-called fuzzy number.

A lot of the measurement is done by measurement equipments, where the result is displayed as a pointer on a scale, digital number, or light symbol on a screen. All these results of measurement are containing different kinds of uncertainty: Variability, errors, and fuzziness. Variability and errors are usually modelled by probability distributions, whereas fuzziness can be represented by so-called *fuzzy numbers*.

2. FUZZY NUMBERS

A single measurement result has to be mathematically described. The most up to date mathematical model are fuzzy numbers.

Definition 1 A fuzzy number x^* is defined by its so-called *characterizing function* $\xi(\cdot)$, which is a real function of one real variable x obeying the following:

1. $0 \leq \xi(x) \leq 1 \qquad \forall x \in \mathbb{R}$
2. The support of $\xi(\cdot)$ is a bounded set, i.e.
$$\text{supp}[\xi(\cdot)] = \{x \in \mathbb{R} \colon \xi(x) > 0\} \subseteq [a, b] \quad \text{with } -\infty < a < b < \infty$$
3. For each $\delta \in (0, 1]$ the so-called δ-Cut $\mathcal{C}_\delta[\xi(\cdot)]$, i.e.
$$\mathcal{C}_\delta[\xi(\cdot)] = \{x \in \mathbb{R} \colon \xi(x) \geq \delta\}$$
is non-empty, and a finite union of compact intervals, i.e.
$$\mathcal{C}_\delta[\xi(\cdot)] = \bigcup_{j=1}^{k_\delta} [a_{j,\delta}, b_{j,\delta}] \quad \text{with} \quad k_\delta \in \mathbb{N}.$$

In Fig. 1. some examples of characterizing functions are depicted. For more details about fuzzy numbers see [4].

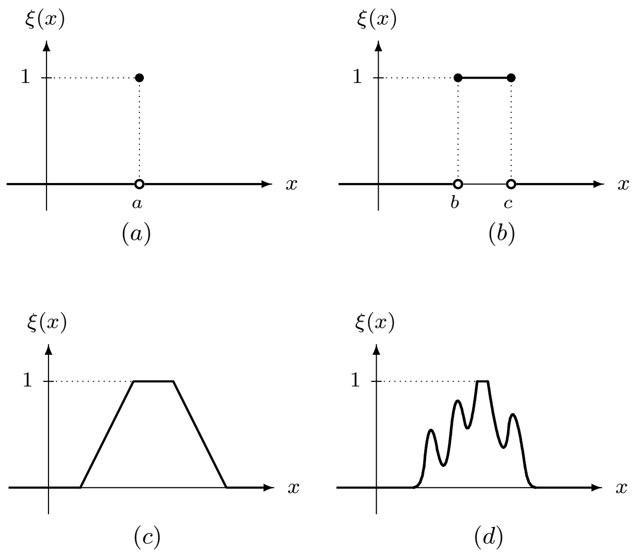

Fig. 1. Examples of characterizing functions.

A real number a in the classical sense is a special type of fuzzy number with the following characterizing function:
$$\xi(\cdot) = \mathbb{1}_{\{a\}}(\cdot),$$
where the function $\mathbb{1}_B(\cdot)$ is called *indicator function* of the set B, defined as:
$$\mathbb{1}_B(x) = \left\{ \begin{array}{ll} 1 & \text{for } x \in B \\ 0 & \text{for } x \notin B \end{array} \right\}$$

3. FUZZY VECTORS

Multidimensional measurement results have also to be mathematically described. Similarly to the one-dimensional case the most up to date mathematical model are fuzzy vectors.

Definition 2 A fuzzy vector $\underline{x}^*$ is defined by its so-called *vector-characterizing function* $\zeta(\cdot)$, which is a real function of k real variables $x_1, \ldots, x_k$ obeying the following:

1. $0 \leq \zeta(x_1, \ldots, x_k) \leq 1 \qquad \forall (x_1, \ldots, x_k) \in \mathbb{R}^k$
2. The support of $\zeta(\cdot, \ldots, \cdot)$ is a bounded set, where

   $$\operatorname{supp}[\zeta(\cdot, \ldots, \cdot)] = \\ = \left\{(x_1, \ldots, x_k) \in \mathbb{R}^k : \zeta(x_1, \ldots, x_k) > 0\right\}$$
3. For each $\delta \in (0, 1]$ the δ-Cut $\mathcal{C}_\delta[\zeta(\cdot, \ldots, \cdot)]$, i.e.

   $$\mathcal{C}_\delta[\zeta(\cdot, \ldots, \cdot)] = \\ = \{(x_1, \ldots, x_k) \in \mathbb{R}^k : \zeta(x_1, \ldots, x_k) \geq \delta\}$$

   is non-empty, and a finite union of compact connected subsets of $\mathbb{R}^k$

In Fig. 2. an example of a vector-characterizing function is depicted. For more details about fuzzy vectors see [4].

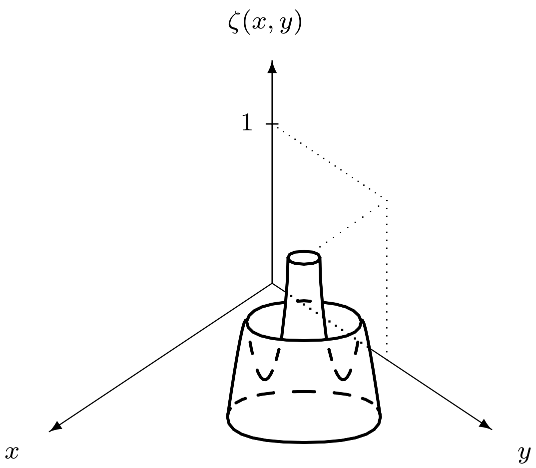

Fig. 2. An example of a vector-characterizing function.

A fuzzy number is a special case of fuzzy vector, it is a one-dimensional fuzzy vector.

4. VECTOR OF FUZZY NUMBERS

Repeated measurement results can be mathematically described by vectors of fuzzy numbers.

Definition 3 A n-dimensional *vector of fuzzy numbers* $(x_1^*, \ldots, x_n^*)$ is a vector containing n fuzzy numbers $x_1^*, \ldots, x_n^*$. It is determined by n characterizing functions $\xi_1(\cdot), \ldots, \xi_n(\cdot)$ belonging to the fuzzy numbers $x_1^*, \ldots, x_n^*$.

We can construct a fuzzy vector from a vector of fuzzy numbers by using a so-called *triangular norm*, or shortly *t*-norm.

Definition 4 A function $T : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is called *t-norm*, if and only if $\forall\, x, y, z \in [0, 1]$ the following conditions are fulfilled:

1. $T(x, y) = T(y, x)$ (T is commutative)
2. $T(T(x, y), z) = T(x, T(y, z))$ (T is associative)
3. $T(x, 1) = x$ (1 is neutral to T)
4. $x \leq y \Rightarrow T(x, z) \leq T(y, z)$ (T is monotone)

For a vector of two fuzzy numbers (x_1^*, x_2^*), where the fuzzy numbers x_1^* and x_2^* have corresponding characterizing functions $\xi_1(\cdot)$ and $\xi_2(\cdot)$, a fuzzy vector $(x_1, x_2)^*$ is given by its vector-characterizing function $\zeta_{(x_1,x_2)^*}(\cdot, \cdot)$, whose values $\zeta_{(x_1,x_2)^*}(x, y)$ are defined based on a t-norm T by

$$\zeta_{(x_1,x_2)^*}(x, y) = T(\xi_1(x), \xi_2(y)) \qquad \forall\, (x, y) \in \mathbb{R}^2.$$

In the general case for a vector of fuzzy numbers $(x_1^*, \ldots, x_n^*)$, where fuzzy numbers $x_1^*, \ldots, x_n^*$ have corresponding characterizing functions $\xi_1(\cdot), \ldots, \xi_n(\cdot)$, a fuzzy vector $(x_1, \ldots, x_n)^*$ is given by its vector-characterizing function $\zeta_{(x_1,\ldots,x_n)^*}(\cdot, \ldots, \cdot)$ whose values are defined based on a t-norm T by using its associativity

$$\zeta_{(x_1,\ldots,x_n)^*}(x_1, \ldots, x_n) = \\ = T(\xi_1(x_1), T(\ldots, T(\xi_{n-1}(x_{n-1}), \xi_n(x_n)) \ldots)) \\ \forall\, (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

A very important *t*-norm is the minimum *t*-norm defined by the following formulae:

$$T_{\min}(x, y) = \min\{x, y\} \qquad \forall\, (x, y) \in [0, 1]^2$$

In Fig. 3. an example of the combination of two fuzzy numbers, using the minimum *t*-norm, is given.

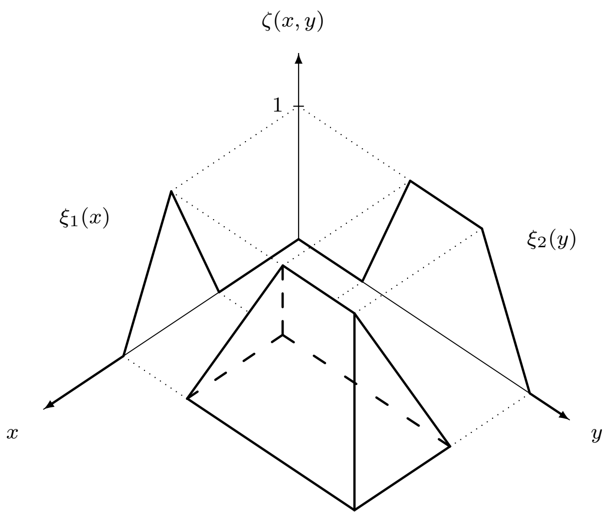

Fig. 3. Combination of two fuzzy numbers with characterizing functions $\xi_1(\cdot)$ and $\xi_2(\cdot)$ based on the minimum *t*-norm.

5. CONSTRUCTING THE CHARACTERIZING FUNCTION OF A FUZZY NUMBER

It is important to discuss how to construct the characterizing function of a fuzzy number from the measurement result. In the following some typical measurement situations will be discussed.

A digital measurement equipment displays the result x as a decimal number containing only finitely many digits. For the infinitely many missing digits all 10 symbols are possible. Denoting by x_0 the real number extending the number x by replacing all the missing digits with 0, and denoting by x_9 the real number extending the number x by replacing all the missing digits with 9, then the resulting interval $[x_0, x_9]$ is a special fuzzy number x^*, whose characterizing function is the indicator function of $[x_0, x_9]$, i.e.

$$\xi(\cdot) = \mathbb{1}_{[x_0,x_9]}(\cdot).$$

For a measurement equipment containing a pointer the best we can do is to make a photo and consider it as a picture of gray intensities. The measurement result can be described by a fuzzy number, whose characterizing function is obtained by normalization of the gray intensity $h(x)$ in the following way:

$$\xi(x) = \frac{h(x)}{\max\{h(y)\colon y \in \mathbb{R}\}} \qquad \forall x \in \mathbb{R}$$

For a measurement equipment, where the result is characterized by the fuzzy boundary of a color intensity picture, the color intensity $g(x)$ can be used to generate the characterizing function $\xi(\cdot)$ of the observed quantity in the following way:

The so-called *scaled rate of change* of the color intensity transition is used, i. e. the normalized derivative of the function $g(\cdot)$, where the derivative is denoted by $g'(\cdot)$:

$$\xi(x) = \frac{|g'(x)|}{\max\{|g'(y)|\colon y \in \mathbb{R}\}} \qquad \forall x \in \mathbb{R}$$

6. COMPUTING WITH FUZZY NUMBERS

The extension principle generalizes classical functions from an arbitrary set M to a second set N for fuzzy elements of M to fuzzy elements in N. Fuzzy numbers and fuzzy vectors are special cases of fuzzy elements.

Let $g: M \to N$ be a classical function. For a fuzzy element A^* in M with membership function $\xi(\cdot)$ the generalized value $g(A^*)$ has to be defined in reasonable way such that it is a fuzzy element in N. In order to obtain the membership function $\eta(\cdot)$ which characterizes the fuzzy element $B^* = g(A^*)$ in N, the values $\eta(\cdot)$ are defined in the following way:

$$\eta(b) = \left\{ \begin{array}{ll} \sup\left\{\xi(a)\colon a \in g^{-1}(\{b\})\right\} & \text{if } g^{-1}(\{b\}) \neq \emptyset \\ 0 & \text{if } g^{-1}(\{b\}) = \emptyset \end{array} \right\} \qquad \forall b \in N$$

By this definition a membership function $\eta(\cdot)$ of a fuzzy element in N is obtained.

For example, let x_1^* and x_2^* be fuzzy numbers with corresponding characterizing functions $\xi_1(\cdot)$ and $\xi_2(\cdot)$, and suppose that we want to add this two fuzzy numbers. The function for the sum of two numbers is

$$g(x, y) = x + y \qquad \forall\,(x, y) \in \mathbb{R}^2.$$

Firstly we create a vector of fuzzy numbers (x_1^*, x_2^*). In the next step we construct a fuzzy vector $(x_1, x_2)^*$ with vector-characterizing function $\zeta(\cdot,\cdot)$ from the characterizing functions of the fuzzy numbers x_1^* and x_2^* by applying the minimum *t*-norm

$$\zeta(x, y) = \min\{\xi_1(x), \xi_2(y)\} \qquad \forall\,(x, y) \in \mathbb{R}^2.$$

Now we can apply the extension principle for the addition of two real numbers and the fuzzy vector $(x_1, x_2)^*$:

$$\eta(z) = \left\{ \begin{array}{ll} \sup\left\{\zeta(x, y)\colon x + y = z\right\} & \text{if } \exists z \in \mathbb{R} : x + y = z \\ 0 & \text{if } \nexists z \in \mathbb{R} : x + y = z \end{array} \right\} \qquad \forall z \in \mathbb{R}$$

The result of applying the extension principle is the above defined characterizing function $\eta(\cdot)$ describing a fuzzy number $y^* = g((x_1, x_2)^*)$. An example of the addition of two fuzzy numbers is given in Fig. 4.

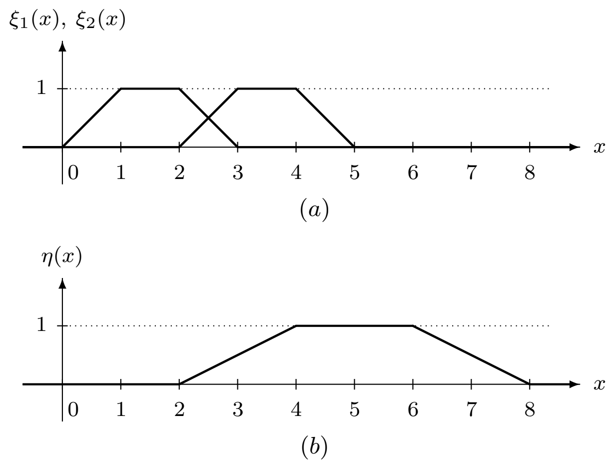

Fig. 4. Characterizing functions $\xi_1(\cdot)$ and $\xi_2(\cdot)$ of fuzzy numbers x_1^* and x_2^* and characterizing function $\eta(\cdot)$ of their sum described by a fuzzy number $y^* = g((x_1, x_2)^*)$.

In the same way the generalized sum of n fuzzy numbers can be computed. The formula for the sum of n numbers is

$$g_{\text{sum}}(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i \qquad \forall\,(x_1, \ldots, x_n) \in \mathbb{R}^n.$$

Let $x_1^*, \ldots, x_n^*$ be fuzzy numbers with corresponding characterizing functions $\xi_1(\cdot), \ldots, \xi_n(\cdot)$. We construct the fuzzy vector $(x_1,\dots,x_n)^*$ with the vector-characterizing function $\zeta(\cdot,\dots,\cdot)$ from the characterizing functions of the fuzzy numbers $x_1^*,\dots,x_n^*$ by applying the minimum *t*-norm

$$\zeta(x_1,\dots,x_n) = \min\{\xi_1(x_1),\dots,\xi_n(x_n)\} \quad \forall\,(x_1,\dots,x_n)\in\mathbb{R}^n.$$

Now we can apply the extension principle for the addition of n real numbers and the fuzzy vector $(x_1,\dots,x_n)^*$:

$$\eta(z) = \begin{cases} \sup\{\zeta(x_1,\dots,x_n)\colon \sum_{i=1}^n x_i = z\} & \text{if } \exists z\in\mathbb{R} : \sum_{i=1}^n x_i = z \\ 0 & \text{if } \nexists z\in\mathbb{R} : \sum_{i=1}^n x_i = z \end{cases} \quad \forall z\in\mathbb{R}$$

The result is a characterizing function $\eta(\cdot)$ describing the fuzzy number $y^* = g_{\text{sum}}((x_1,\dots,x_n)^*)$.

For more explanation of the extension principle see the monograph [4].

7. FUZZY MEAN AND FUZZY STANDARD DEVIATION

In classical measurement analysis the mean value and the standard deviation of repeated measurements $x_1,\dots,x_n$ are taken.

$$x_{\text{mean}} = \frac{1}{n}\sum_{i=1}^n x_i$$

$$x_{\text{st.dev}} = \sqrt{\frac{1}{n-1}\sum_{i=1}^n [x_i - x_{\text{mean}}]^2}$$

For a vector of fuzzy numbers $(x_1^*,\dots,x_n^*)$, where the fuzzy numbers $x_1^*,\dots,x_n^*$ have corresponding characterizing functions $\xi_1(\cdot),\dots,\xi_n(\cdot)$, this operation has to be generalized. This is possible based on the extension principle.

In this paper an application of the extension principle for computing the fuzzy mean value and the fuzzy standard deviation using the minimum *t*-norm is given.

Let $x_1^*,\dots,x_n^*$ be fuzzy numbers with corresponding characterizing functions $\xi_1(\cdot),\dots,\xi_n(\cdot)$. Then the fuzzy mean value x^*_{mean} and the fuzzy standard deviation $x^*_{\text{st.dev}}$ have characterizing functions $\xi_{\text{mean}}(\cdot)$ and $\xi_{\text{st.dev}}(\cdot)$ respectively defined in the following way:

$$\xi_{\text{mean}}(y) = \begin{cases} \sup\left\{\zeta(\underline{x}_n)\colon \underline{x}_n \in g^{-1}_{\text{mean}}(\{y\})\right\} & \text{if } g^{-1}_{\text{mean}}(\{y\}) \neq \emptyset \\ 0 & \text{if } g^{-1}_{\text{mean}}(\{y\}) = \emptyset \end{cases} \quad \forall y\in\mathbb{R}$$

$$\xi_{\text{st.dev}}(y) = \begin{cases} \sup\left\{\zeta(\underline{x}_n)\colon \underline{x}_n \in g^{-1}_{\text{st.dev}}(\{y\})\right\} & \text{if } g^{-1}_{\text{st.dev}}(\{y\}) \neq \emptyset \\ 0 & \text{if } g^{-1}_{\text{st.dev}}(\{y\}) = \emptyset \end{cases} \quad \forall y\in\mathbb{R}$$

where $\underline{x}_n = (x_1,\dots,x_n)$ is a vector of real numbers, and the vector-characterizing function $\zeta(\cdot,\dots,\cdot) : \mathbb{R}^n \to [0,1]$ is constructed by using the minimum *t*-norm:

$$\zeta(x_1,\dots,x_n) = \min\{\xi_1(x_1),\dots,\xi_n(x_n)\} \quad \forall(x_1,\dots,x_n)\in\mathbb{R}^n$$

The well known functions for computing the mean value $g_{\text{mean}}(\cdot,\dots,\cdot) : \mathbb{R}^n \to \mathbb{R}$ and computing the standard deviation $g_{\text{st.dev}}(\cdot,\dots,\cdot) : \mathbb{R}^n \to \mathbb{R}$ are defined in the following way:

$$g_{\text{mean}}(x_1,\dots,x_n) = \frac{1}{n}\sum_{i=1}^n x_i \qquad \forall(x_1,\dots,x_n)\in\mathbb{R}^n$$

$$g_{\text{st.dev}}(x_1,\dots,x_n) = \sqrt{\frac{1}{n-1}\sum_{i=1}^n \left[x_i - \left(\tfrac{1}{n}\textstyle\sum_{j=1}^n x_j\right)\right]^2} \quad \forall(x_1,\dots,x_n)\in\mathbb{R}^n$$

8. CONCLUSIONS

Realistic results of measurements are usually not precise numbers, but they are more or less non-precise. The kinds of uncertainty are variability, errors and fuzziness. In this paper we are dealing with fuzziness, which can be expressed by a fuzzy number.

The introduction to the theoretical background of fuzzy numbers is presented. Methods on how to construct a fuzzy number from a measurement result are given. Furthermore generalized (fuzzy) statistics containing fuzzy mean value and fuzzy standard deviation are explained.

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