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BAYESIAN ANALYSIS OF REPEATED MEASUREMENTS AFFECTED BY A SYSTEMATIC ERROR AND ITS APPLICATION TO CONFORMITY ASSESSMENT

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Abstract **-** The Bayesian analysis of a series of correlated indications of an unknown quantity is here presented when they are modelled by a joint Gaussian distribution and their covariance is assumed to be the (known) squared uncertainty associated with a systematic effect common to all the indications. An interesting application of the obtained results to the conformity assessment of a series production is also presented. A criterion is derived so that at least a portion p_1 of the series production shows to have a characteristic value below a prescribed limit, with a probability not less than p_2.

Keywords: Bayesian inference, repeated measurements, systematic effect, conformity assessment.

1. INTRODUCTION

The Bayesian treatment of a series of n independent indications of an unknown quantity Q, regarded as modelled by a Gaussian distribution with unknown expected value μ (the parameter to be inferred) and standard deviation σ, is a well stabilized tool within the metrological framework (see [1], Clause 6.4.9.2). Considering non-informative prior distributions for the parameters of the Gaussian distribution and using Bayes' theorem, the marginal probability density function (pdf) for μ is a scaled and shifted Student t-distribution with $n-1$ degrees of freedom.

In general, one of the most important advantages of the application of Bayesian analysis to measurement problems is the unified and coherent treatment of the uncertainty contributions arising from random and systematic effects [2]. Nonetheless, few applications of the Bayesian uncertainty analysis to actual measurement problems involving systematic effects are available. In a recently published "GUM anniversary issue" of Metrologia, including [2, 3, 4] among the other papers, the role of Bayesian analysis is compared with the procedures described in the "Guide to the Expression of Uncertainty in Measurement" (GUM) [5] and its supplements for the evaluation of measurement uncertainty. In [3], a simple example is provided where a series of repeated indications is obtained by using a measurement device affected by a systematic bias due to imperfect calibration. Such work belongs to a vein of previously published papers (see, for example, [6] and [7]) aimed at studying and comparing the results obtained with pure Bayesian inference with those obtained by application of the Supplements to the GUM to the general measurement model $Y = f(X, Z)$, where Y is the measurand, X is an input quantity for which normally distributed observations are available ('type A' variable) and Z is another input quantity for which a proper pdf is available ('type B' variable). In all those works, indications are invariably modelled by independent and identically distributed Gaussian random variables. This assumption, which is not obvious because of the presence of a systematic effect Z which should correlate the indications, is justified by taking into account, within the Bayesian framework, that if the systematic error is given then the (conditional) pdfs for the repeated indications are actually independent although biased by the systematic error.

An alternative Bayesian analysis is here developed based on deductive reasoning and the use of a multivariate distribution for the indications which takes automatically into account the correlation between them. The aim is to derive the posterior joint pdf for parameters μ and σ of the Gaussian distribution of a quantity Q, whose indications are all correlated by a common systematic effect for which the associated uncertainty is supposed to be known. Once such pdf is obtained, all the desired results can be derived, such as the marginal pdfs of the two parameters and their first and second moments.

In this framework, an interesting application to the assessment of compliance of a series production with a certain specification limit is elaborated, taking measurement uncertainty into account. For example, a criterion is derived so that at least a portion p_1 of the series production shows to have a characteristic Q value below a prescribed limit L, with a probability not less than p_2.

2. BAYESIAN ANALYSIS

2.1. Measurement model

The considered model is

$$q_{e_i} = q_i + e, \qquad (1)$$

for $i = 1, \ldots, n$, where q_{e_i} are the indication values of a quantity Q_e, q_i are realizations of an unknown quantity Q, characterized by a pdf with mean value μ and standard deviation σ, and e is the realization of a systematic effect E due to the calibration of the measurement device. In the present treatment, the systematic effect is supposed to be corrected, hence its expected value is zero, whereas its standard uncertainty u_e is assumed to be known. The model between the corresponding quantities is then

$$Q_{e_i} = Q_i + E. \tag{2}$$

Within a multivariate context, the indication vector $\boldsymbol{Q_e} = (Q_{e_1}, Q_{e_2}, \cdots, Q_{e_n})^\top_{(n\times 1)}$ is modelled by a multivariate normal distribution $\boldsymbol{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, with mean vector $\boldsymbol{\mu} = \mu\mathbf{1}$, with $\mathbf{1} = (1,\ 1, \cdots, 1)^\top_{(n\times 1)}$, and covariance matrix $\boldsymbol{\Sigma}_{(n\times n)}$

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma^2 + u_e^2 & u_e^2 & \cdots & u_e^2 \\ u_e^2 & \sigma^2 + u_e^2 & \cdots & u_e^2 \\ \vdots & \vdots & \ddots & \vdots \\ u_e^2 & u_e^2 & \cdots & \sigma^2 + u_e^2 \end{bmatrix}. \tag{3}$$

Notice that u_e^2 covariates all the indications and is perfectly known, whereas the inference is focused just on parameters μ and σ. Further, if Q_i and E in (2) are independent one each other, then variable $\boldsymbol{Q_e}$ is jointly normally distributed **iff** Q_i and E are both normally distributed (for each i).

The corresponding likelihood $l(\mu, \sigma|\boldsymbol{q_e})$, seen as the conditional pdf of the (vector) indication values $\boldsymbol{q_e} = (q_{e_1}, q_{e_2}, \cdots, q_{e_n})^\top_{(n\times 1)}$ given the parameter values μ and σ, is then the joint Gaussian distribution:

$$l(\mu, \sigma|\boldsymbol{q_e}) = \frac{1}{\sqrt{(2\pi)^n|\boldsymbol{\Sigma}|}} \exp\left[-\frac{1}{2}(\boldsymbol{q_e} - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1}(\boldsymbol{q_e} - \boldsymbol{\mu})\right]. \tag{4}$$

The considered (non-informative) joint prior pdf for the parameters is

$$f_0(\mu, \sigma) = f_0(\mu) f_0(\sigma) \propto 1/\sigma. \tag{5}$$

2.2. Joint and marginal posterior pdfs

According to the Bayes' theorem, the posterior pdf $f(\mu, \sigma|\boldsymbol{q_e})$ is given by

$$f(\mu, \sigma|\boldsymbol{q_e}) \propto l(\mu, \sigma|\boldsymbol{q_e}) f_0(\mu, \sigma). \tag{6}$$

Under assumptions (4) and (5), it was derived (see the Appendix) the analytical expression for (6), resulting equal to

$$\begin{aligned} f(\mu, \sigma|\boldsymbol{q_e}) = \;& \frac{\frac{2}{s}\left(\frac{s}{\sigma}\right)^n \left(\frac{n-1}{2}\right)^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)} \exp\left[-\frac{1}{2}\frac{(n-1)s^2}{\sigma^2}\right] \cdot \\ & \frac{1}{\sqrt{2\pi}\sqrt{\sigma^2/n + u_e^2}} \exp\left[-\frac{1}{2}\frac{(\overline{q_e} - \mu)^2}{\sigma^2/n + u_e^2}\right], \end{aligned} \tag{7}$$

where $\overline{q_e} = \frac{1}{n}\sum_{i=1}^n q_{e_i}$ and $s^2 = \frac{1}{n-1}\sum_{i=1}^n (q_{e_i} - \overline{q_e})^2$ are the sample mean and the sample variance, respectively, calculated on the indication values $\boldsymbol{q_e}$.

Integrating (7) with respect to σ and μ, marginal pdfs $f(\mu|\boldsymbol{q_e})$ and $f(\sigma|\boldsymbol{q_e})$ can be obtained, respectively:

$$\begin{aligned} f(\mu|\boldsymbol{q_e}) = \;& \frac{\frac{2}{s}\left(\frac{n-1}{2}\right)^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)} \int_0^\infty \frac{\left(\frac{s}{\sigma}\right)^n}{\sqrt{2\pi}\sqrt{\sigma^2/n + u_e^2}} \exp\left[-\frac{1}{2}\frac{(n-1)s^2}{\sigma^2}\right] \\ & \exp\left[-\frac{1}{2}\frac{(\overline{q_e} - \mu)^2}{\sigma^2/n + u_e^2}\right] d\sigma, \end{aligned} \tag{8}$$

$$f(\sigma|\boldsymbol{q_e}) = \frac{\frac{2}{s}\left(\frac{s}{\sigma}\right)^n \left(\frac{n-1}{2}\right)^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)} \exp\left[-\frac{1}{2}\frac{(n-1)s^2}{\sigma^2}\right]. \tag{9}$$

It was analytically demonstrated that (8) is the pdf of the difference between a scaled and shifted Student t-distribution with $\nu = n - 1$ degrees of freedom (shifted by $\overline{q_e}$ and scaled by $s/\sqrt{n}$, as in the well-known case of independent indications) and a normal distribution $N(0, u_e^2)$, its expected value and variance being, hence, respectively

$$E(\mu|\boldsymbol{q_e}) = \overline{q_e}, \tag{10}$$

$$V(\mu|\boldsymbol{q_e}) = \frac{n-1}{n-3}\frac{s^2}{n} + u_e^2. \tag{11}$$

Expression (11) is in agreement with the common metrological sense of a noisy contribution in the experiment, being reduced by increasing the number of repeated observations, and a systematic term which cannot be reduced be repeating the experiment.

Expression (9), seen as the pdf of random variable σ^2, is the pdf of a scaled inverse chi-squared distribution with $\nu = n - 1$ degrees of freedom, with expected value and variance equal to

$$E(\sigma^2|\boldsymbol{q_e}) = \frac{n-1}{n-3}s^2, \tag{12}$$

$$V(\sigma^2|\boldsymbol{q_e}) = \left(\frac{n-1}{n-3}\right)^2 \frac{2s^4}{n-5} = \frac{2}{n-5}\, E^2(\sigma^2|\boldsymbol{q_e}). \tag{13}$$

2.3. Comparison with the case of independent indication values

It can be noted that the joint posterior (7) is the product of two factors: the same scaled inverse chi-squared pdf $f(\sigma|\boldsymbol{q_e})$ which is defined in (9) and a normal pdf with mean $\overline{q_e}$ and variance $\sigma^2/n + u_e^2$. Moreover, systematic uncertainty u_e affects only the latter factor, and expression (7) reduces, for $u_e = 0$, i.e., for independent Gaussian observations, to the better-known joint posterior pdf for μ and σ (see, for example, expression (6.12) in [8]).

Hence, since the joint pdf can be also written as

$$f(\mu, \sigma|\boldsymbol{q_e}) = f(\mu|\sigma, \boldsymbol{q_e}) f(\sigma|\boldsymbol{q_e}), \tag{14}$$

it follows that the conditional pdf $f(\mu|\sigma, \boldsymbol{q_e})$ is the pdf of a normal distribution $N(\overline{q_e}, \sigma^2/n)$, when the indications are independent, or the pdf of a normal distribution $N(\overline{q_e}, \sigma^2/n + u_e^2)$, when the indications are correlated. Notice that the latter normal distribution can be seen as given by the difference of two independent normal distributions: $N(\overline{q_e}, \sigma^2/n)$ and $N(0, u_e^2)$.

Therefore, the following parallelism holds for the marginal pdf of μ in the case of independent or correlated indication values, respectively:

- When $f(\mu|\sigma, \boldsymbol{q_e})$ is the pdf of a $N(\overline{q_e}, \sigma^2/n)$ $\Rightarrow$ $f(\mu|\boldsymbol{q_e})$ is the pdf of the scaled and shifted t-distribution $\overline{q_e} + s/\sqrt{n}\, T_{n-1}$ (T_{n-1} indicating a random variable with a Student t-distribution with $n-1$ degrees of freedom);

- When $f(\mu|\sigma, \boldsymbol{q_e})$ is the pdf of the difference between $N(\overline{q_e}, \sigma^2/n)$ and $N(0, u_e^2)$ $\Rightarrow$ $f(\mu|\boldsymbol{q_e})$ is the pdf of the random variable $\overline{q_e} + s/\sqrt{n}\, T_{n-1} - N(0, u_e^2)$. Notice that the same result would be obtained in the framework of [3, 6, 7] when considering the measurand model $Y = X - Z$.

3. APPLICATION TO CONFORMITY ASSESSMENT

In this section, a way to determine a criterion for assessing the compliance of a series production with a desired specification limit is shown. Thanks to the results presented in the previous section, a criterion is derived so that at least a portion p_1 of the series production shows to have a characteristic Q value below a prescribed limit L, with a probability not less than p_2. In the field of electromagnetic compatibility, for example, document [9] prescribes the "80 %/80 % rule" as a criterion for conformity assessment of series production requiring that at least 80 % of the products comply with the emission limit with a probability of not less than 80 %.

The production is considered as modelled by a normal random variable Q, with mean value μ and standard deviation σ; the value of the i-th product is q_i. Let us suppose that such value cannot be directly observed, but that only q_{e_i} values can be obtained, which are all affected by a common systematic error due to a systematic effect in the employed measuring instrument. The systematic error is corrected by means of the instrument calibration factor/curve in order to have a null expected value, but the associated calibration uncertainty u_e remains not negligible.

Model (2) is therefore appropriate for treating this problem. Given the values of μ and σ, the fraction p_1 of the production satisfing the (upper) limit prescription value L is

$$P(Q < L|\mu, \sigma) = P\left(\frac{Q-\mu}{\sigma} < \frac{L-\mu}{\sigma}|\mu, \sigma\right) = p_1. \tag{15}$$

Hence, whenever $(L-\mu)/\sigma \geq z_{p_1}$, where z_{p_1} is the p_1^{th} quantile of a standard normal distribution, probability (15) is larger or equal to p_1.

However, according to the Bayesian view, μ and σ are random variables, the state of knowledge about which is encoded by their joint pdf (7). Hence, the requirement is that the probability of $(L-\mu)/\sigma$ being larger than z_{p_1} should be at least equal to p_2. Therefore, the scope is to determine the limit value L so that

$$P\left(\frac{L-\mu}{\sigma} > z_{p_1}|\boldsymbol{q_e}\right) = P\left(\mu + z_{p_1}\sigma < L|\boldsymbol{q_e}\right) \geq p_2. \tag{16}$$

Let L_{p_2} be the limit value such that $P\left(\mu + z_{p_1}\sigma < L_{p_2}|\boldsymbol{q_e}\right) = p_2$, then L_{p_2} satisfies the following equation:

$$\int_0^{\infty}\int_{-\infty}^{L_{p_2}-z_{p_1}\sigma} f(\mu, \sigma|\boldsymbol{q_e})\, d\mu\, d\sigma = p_2. \tag{17}$$

Therefore, for any given probabilty values p_1 and p_2, the corresponding limit value L_{p_2} can be numerically determined by calculating integral (17), which is the integral of the joint pdf (7) on the shaded region plotted in Fig. 1. Consequently, relation (16) holds for any $L \geq L_{p_2}$.

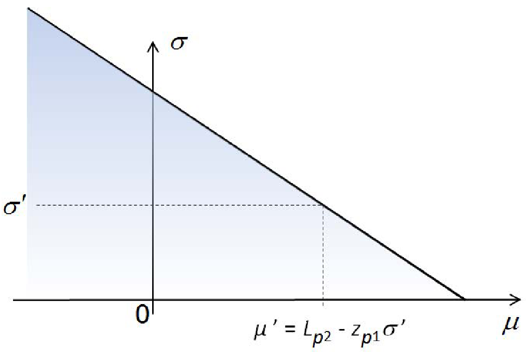

Fig. 1. Domain of integration of the joint pdf $f(\mu, \sigma|\boldsymbol{q_e})$.

It can be shown that

$$L_{p_2} = k_{p_2} s + \overline{q_e}, \tag{18}$$

where k_{p_2} is a function of n and s/u_e. As an example, Table 1 shows some k_{p_2} values calculated for several values of n and s/u_e, when $p_1 = p_2 = 0.8$.

Notice that for $s/u_e = \infty$, i.e., when $u_e = 0$, the obtained k_{p_2} values are identical to those actually prescribed by the "80 %/80 % rule" as reported in clause 5.1 of [9].

The resulting conformity criterion, relying on the whole available information about the series production, is then a relationship between probabilities p_1 and p_2 and limit L: given two of the three ingredients, the third one can be always obtained.

4. CONCLUSIONS

The Bayesian analysis of a series of n correlated indication values of an unknown quantity Q was presented in the case of indications distributed according to a joint Gaussian distribution $\boldsymbol{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, with mean vector $[\mu, \mu, \ldots, \mu]^T$ and covariance matrix $\boldsymbol{\Sigma}$ (3), in which σ accounts for the noisy contribution to the standard deviation of each Q_{e_i}, while u_e^2, that is the covariance between

Table 1. Values of k_{p_2} obtained for several values of n and s/u_e, when $p_1 = p_2 = 0.8$.

k_{p_2} values		n					
		2	5	10	20	50	100
s/u_e	∞	3.42	1.51	1.24	1.10	0.99	0.95
	10	3.43	1.52	1.25	1.11	1.02	0.98
	1	3.71	2.04	1.83	1.75	1.71	1.69
	0.3	5.72	3.93	3.74	3.69	3.66	3.66
	0.1	11.8	9.50	9.34	9.30	9.27	9.27

Q_{e_i} and Q_{e_j}, with $j \neq i$, is the squared uncertainty associated with a common systematic effect (due, for example, to the calibration term of the utilized measurement instrument). Assuming non-informative prior distributions for the parameters of interest, μ and σ, their joint posterior pdf was derived, as well as, in particular, the marginal density for μ, resulting in the pdf of the difference between a scaled and shifted t-distribution and a normal distribution with zero mean and variance u_e^2.

The same marginal pdf for μ would be obtained in the framework of [3, 6, 7] also. With respect to those works, however, the Bayesian analysis here reported differs in that it directly starts from jointly distributed (hence correlated) indications, whereas the above-mentioned papers relies on measurand model of type $Y = f(X, Z)$ in which the systematic effect Z is modelled as separated from the indications X and, therefore, the indications are mutually independent.

Moreover, the present work openly provides the analytical expression for the joint posterior pdf for μ and σ. Such pdf was used for application to the conformity assessment of series production in the electromagnetic compatibility field. A criterion was derived so that at least a portion p_1 of the series production shows to have a characteristic value below a prescribed limit, with a probability not less than p_2. As a confirmation of the validity of the obtained results, they perfectly adhere, in the case $u_e = 0$, to those provided by the formulae presently applied in the field.

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APPENDIX

By means of symbolic analysis, determinant $|\boldsymbol{\Sigma}|$ of covariance matrix (3) can be written as

$$|\boldsymbol{\Sigma}| = \sigma^{2n-2}(\sigma^2 + n u_e^2). \tag{19}$$

Rearranging also term $(\boldsymbol{q_e} - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1} (\boldsymbol{q_e} - \boldsymbol{\mu})$ within expression (4), one has that

$$(\boldsymbol{q_e} - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1} (\boldsymbol{q_e} - \boldsymbol{\mu}) = \frac{(n-1)s^2}{\sigma^2} + \frac{(\overline{q_e} - \mu)^2}{\sigma^2/n + u_e^2}, \tag{20}$$

where $\overline{q_e} = \frac{1}{n}\sum_{i=1}^{n} q_{e_i}$ and $s^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(q_{e_i} - \overline{q_e}\right)^2$ are the sample mean and the sample variance, respectively, calculated on the indication values $\boldsymbol{q_e}$. Substituting (19) and (20) into (4), one gets

$$l(\mu, \sigma|\boldsymbol{q_e}) = \frac{1}{\sqrt{(2\pi)^n(\sigma^2+nu_e^2)\sigma^{n-1}}} \exp\left[-\frac{1}{2}\frac{(n-1)s^2}{\sigma^2}\right] \exp\left[-\frac{1}{2}\frac{(\overline{q_e}-\mu)^2}{\sigma^2/n+u_e^2}\right]. \tag{21}$$

Therefore, substituting (5) and (21) into (6), one gets

$$f(\mu, \sigma|\boldsymbol{q_e}) \propto \frac{1}{\sigma^n\sqrt{\sigma^2/n+u_e^2}} \exp\left[-\frac{1}{2}\frac{(n-1)s^2}{\sigma^2}\right] \exp\left[-\frac{1}{2}\frac{(\overline{q_e}-\mu)^2}{\sigma^2/n+u_e^2}\right]. \tag{22}$$

It is possible to analytically determine the constant of proportionality K such that

$$\int_0^\infty \int_{-\infty}^\infty \frac{K}{\sigma^n\sqrt{\sigma^2/n+u_e^2}} \exp\left[-\frac{1}{2}\frac{(n-1)s^2}{\sigma^2}\right] \exp\left[-\frac{1}{2}\frac{(\overline{q_e}-\mu)^2}{\sigma^2/n+u_e^2}\right] d\mu\, d\sigma = 1. \tag{23}$$

At the first step, double integral in (23) is solved with respect to μ as follows:

$$\int_{-\infty}^\infty \frac{K}{\sigma^n\sqrt{\sigma^2/n+u_e^2}} \exp\left[-\frac{1}{2}\frac{(n-1)s^2}{\sigma^2}\right] \exp\left[-\frac{1}{2}\frac{(\overline{q_e}-\mu)^2}{\sigma^2/n+u_e^2}\right] d\mu = \sqrt{2\pi}\frac{K}{\sigma^n}\exp\left[-\frac{1}{2}\frac{(n-1)s^2}{\sigma^2}\right]. \tag{24}$$

Hence, expression (24) is integrated with respect to σ (see [10], Sections 7.4.4 and 7.4.5):

$$\int_0^\infty \sqrt{2\pi}\frac{K}{\sigma^n}\exp\left[-\frac{1}{2}\frac{(n-1)s^2}{\sigma^2}\right] d\sigma = \sqrt{2\pi}K\frac{\Gamma\left(\frac{n-1}{2}\right)}{2\left[\frac{(n-1)s^2}{2}\right]^{\frac{n-1}{2}}}. \tag{25}$$

Now, requiring (25) being equal to 1, the normalizing coefficient K is determined as

$$K = \sqrt{2/\pi}\frac{\left[\frac{(n-1)s^2}{2}\right]^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)}, \tag{26}$$

from which expression (7) is finally obtained.

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